CONTRIBUTIONS TO BIO—FLUID MECHANICS

by

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TO MY

PARENTS AND WIFE

CERTIFICATE

This is to certify that the matter embodied in the thesis entitled "Contributions to Bio-fluid Mechanics" by Mr. Balakavi Raghuprasada Rao for the award of the Degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur, is a record of bonafide research work carried out by him under my supervision and guidance. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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(B. RAGHUPRASADA RAO)

CONTENTS

CHAPTER - I

GENERAL INTRODUCTION

1.1 INTRODUCTION

The term "Bio-fluid mechanics" refers to that study of biological systems where the concepts of fluid mechanics, relevant to physiological flows such as blood flow through the cardiovascular system, flow through reproductive organs, air transport through the bronchial tree etc., are used. Since the beginning of the civilization, man has been trying to unveil the mysteries of the functioning of the human body. In doing so he has used various theories and laws contributed by our great scientists such as Harvey, Euler, Newton, Poiseuille, Borelli, Helmholtz, Boyle, Hooke, Young, Fick etc. (Fung : Appl. Mech. Review (1960)).

Since the physiological systems are very complex, studying them realistically is a very difficult task. Several simplifying assumptions have been made in the past and even now, to study them analytically. That is why in certain areas of physiology the experimental research is much advanced than the corresponding analytical investigation.

In this thesis an attempt has been made to explain the mechanisms and causes of functioning of the following systems by making a mathematical model which takes into account the behaviour of the blood such as its viscosity variation, non-Newtonian character etc.

(i) Blood flow through elastic arteries.

(ii) Swimming of spermatozoa in cervical canal.

(iii) Dispersion of a solute (drug) in a fluid with variable viscosity and with non-Newtonian behaviour.

A brief introduction and survey of these topics are given below so that the work presented in this thesis could be seen in its proper perspective.

1.2 BLOOD FLOW

Harvey (1615) is credited with the discovery of blood circulation which is effected by the pumping of the heart, containing four chambers, two on the right and two on the left known as auricles and ventricles. Blood is pumped to various parts of the body through the arteries from the left half and brought back to the right half of the heart via veins. This is called 'systemic circulation'. The second circulation is called the 'pulmonary circulation' in which the impure blood received from various parts of the body by the right portion of the heart is sent to lungs for purification. From lungs it goes to the left half of the heart for being pumped to various parts of the body again.

Aorta, the main blood transporting tube emanating from the left part of the heart branches out at several points fecilitating blood supply to various parts of the body. All the major blood vessels have the property of longitudinal decrease in the area of cross-section known as 'tapering'. Towards the end of the arterial system, we have

the smallest artery called 'arteriole' which again subdivides into large number of capillaries. In capillary region, the exchange of oxygen and carbon dioxide between the tissues and blood takes place. Capillaries join together to become 'venules' which in turn join to become still larger vessels 'veins'. The veins from the lower portion of the body unite to form 'inferior vena cava' and those from the upper portion of the body join together to form 'superior vena cava'. The general size of all the blood vessels and their wall structure are described in Burton (1965), Lightfoot (1974).

Blood is the transporting fluid which carries with it oxygen and various other nutrients for the body. It is a suspension of red cells, white cells and platelets in plasma. There are about 5×10^9 cells in a milli-litre of blood. About 5% of these are platelets and 1/6% are white cells, Middleman (1972).

Plasma is a fluid in which cells are suspended to form the whole blood. It consists of 90% of water and 7% of proteins, Middleman (1972). The major constituents of plasma proteins are Albumin (4.5% by weight), Globulins (2.5%) and Fibrinogen (0.3%), Lightfoot (1974). It is established and generally accepted that plasma is a Newtonian fluid. Careful tests have been conducted to determine its viscosity over a wide range of shear rates, 0.1-1200 s^{-1} and it is found that it behaves like a Newtonian fluid, Copley and Scott-Blair (1960), Charm and Kurland (1962), Cokelet et. al. (1963) and Merrill et. al. (1964, 1965). There are, however, some reports

regarding the non-Newtonian behaviour of plasma but McDonald (1974) attributes this to the presence of denatured protein layer at the interface of plasma and air in the viscometer.

Red cell is a flexible bi-concave disc having a radius of about 4.5μ and thickness 2.5μ at the periphery and 1.0μ at the centre, Burton (1965), Lightfoot (1974) and Middleman (1972). Its main function is to carry oxygen from lungs to tissues and get carbon dioxide from tissues to the lungs. The volume percentage of red cells present in the blood is called 'haemotocrit'. For a normal healthy individual the haemotocrit value willbe about 45%. It is found that an increase in haemotocrit causes an increase in the viscosity of blood. At low shear rates red cells form aggregates. A long chain of red cells thus forming an aggregate is called a roulex. Fibrinogen, a plasma protein is responsible for forming roulex of red cells. The actual mechanism with which rouleau are formed is still unknown. The whole blood exhibits non-Newtonian behaviour particularly when it is flowing under low shear rates. Several models have been suggested for the non-Newtonian behaviour of the blood.

It is suggested that blood behaves like a powerlaw model for low shear rates, Charm and Kurland (1962), Hershey and Cho (1966), Huckaba et. al. (1968). But it has been found that this model does not hold good for shear rates more than 20 s^{-1}, Charm and Kurland (1965).

It is also observed that at low shear rates blood exhibits 'yield stress', Cokelet et. al. (1963). A fluid capable of resisting motion until the shear stress attains a finite value is said to possess 'yield stress'. Reiner and Scott-Blair (1959) suggested that blood behaves like a Casson fluid. It is shown that the Casson equation fits well over a wide range of shear rates 1 - 100,000 s^{-1}, Charm and Kurland (1962, 1965) and 0.1 - 20 s^{-1}, Merril et. al. (1963, 1964), Cokelet et. al. (1963). It is therefore generally accepted that blood behaves like a Casson model fluid, particularly at low shear rates.

In major arteries which are elastic the blood flow is pulsatile. Euler (1775) was the first to consider the blood flow through an elastic tube by considering the blood as an incompressible, inviscid fluid. A comprehensive treatment of wave propagation in a viscous fluid in an elastic pipe can be found in the works of Womersley (1957, 1958) and McDonald (1974). Anisotropy and longitudinal and radial constraints of the wall are considered by Atabek (1968) and Mirsky (1967) in analysing the problem. Cox (1968) investigated the effect of visco-elasticity of the wall on the flow behaviour. Whirlow and Rouleau (1965) discussed the periodic flow of blood through thick-walled elastic tube. Attinger (1964), Middleman (1972) dealt with the pulsatile flow of Newtonian fluid in an elastic pipe. Lightfoot (1974) discussed the pulsatile flow of a Newtonian fluid in an elastic pipe and through a branching circuit. Rodkiewicz et. al. (1971, 1973) experimentally investigated the flow behaviour at a branching point in the case of rigid pipes. Also Ferguson et. al.(1972)

studied experimentally the effect of branching in rigid pipes on flow behaviour of a fluid.

It may be concluded that the blood transport in the cardio-vascular system, in general, depends upon the following, Fung et. al. (1972).

(i) Nature of the blood, its viscosity, its non-Newtonian character, cell concentration and distribution etc.

(ii) Blood vessel, its geometry, size and shape, curvature, branching, tapering etc.; its elasticity (flexibility), porosity etc.

(iii) Nature of the flow such as pulsatile, laminar, turbulent, micro-circulation etc.

In this thesis the blood flow through elastic arteries has been studied by considering its non-Newtonian behaviour. The effects of branching have also been investigated.

1.3 MOTION OF SPERMATOZOA

Spermatozoa are living cells produced in male reproductive system. Each spermatozoon has a head and a tail. Its length is 5×10^{-3} cm and tail diameter is 10^{-5} cm. It can propel itself in forward direction by sending waves of lateral displacement down its tail. Its speed is $50\,\mu$ sec, Smelser et. al. (1974).

During coitus sperm cells are deposited in the vagina. They must move through the mucus filling the cervical canal to reach the

oviduct. As soon as ovum is released from the ovary, it is sucked into oviduct and it starts moving towards uterus. Because of the short life span of the ovum, the fertilisation should occur in oviduct itself. Transport of sperm from vagina to oviduct is too rapid to be accounted for by the motility of the sperm. The role of cervical mucus in helping the sperm to move fast has been a matter of active research and controversy. Several mechanisms for this have been suggested and the work done in this regard has been reviewed, Sobrero (1963), Davajan et. al. (1970), Moghissi (1969, 1971), Moghissi and Blandau (1972), Smelser (1972).

Odeblad (1962) proposed a transport mechanism of the cervical mucus. He studied the physical structure of cervical mucus and found it to be a suspension of macromolecules in a water-like liquid (1959, 1968). Its viscosity is 0.03P, Odeblad (1962). At mid-cycle period these molecules group together and align themselves near the wall of the cervical canal. In the luteal phase, the mucus is like a close mesh. This structure of mucus in mid-cycle period is supported by Elstein et. al. (1971), Davajan et. al. (1971).

Odeblad (1962) suggested that the long chain of molecules which aligned themselves along the canal, vibrate fecilitating quick transport of the sperm.

Taylor (1951) was the first to consider the motion of spermatozoa mathematically. Swimming of micro-organisms in an unbounded medium

was also discussed by Hancock (1953), Gray and Hancock (1955), Brokaw (1970) A small amplitude solution of the swimming of a sperm in a bounded medium was furnished by Reynolds (1965) A long wave length solution was discussed by Shack and Lardner (1974) Considering Odeblad's theory, Smelser et al (1974) investigated the swimming of sperm in an active channel in which the walls of the channel also did vibrate Lighthill (1975) presented a general theory of swimming of micro-organisms at low Reynolds numbers while Pironneau (1975) discussed the motion of uniflagellated micro-organisms using resistive force theory

The actual mechanism of the dynamical interaction of the sperm and the cervical canal is not well understood Here an attempt is made to explain this by using the concepts of lubrication theory, Cameron (1962)

1 4 DISPERSION

The dispersion of soluble matter in laminar flow has biological applications such as drug distribution in the body The diffusion/dispersion process in fluids has been studied by many investigators under steady and unsteady conditions, Crank (1975), Crank and Park (1968), Crank and Gupta (1972), Cumming et al (1966), Gill et al (1970,71) One of the simplest methods to study the dispersion was suggested by Sir Geoffrey Taylor (1953) who investigated the dispersion of soluble matter in a solvent flowing under laminar conditions in a circular tube He found that relative to a plane moving with the mean speed of the flow the solute diffuses with an apparent diffusion

coefficient $R^2U^2/48D$ where R is the radius of the pipe, U is the mean speed and D is the molecular diffusion coefficient Taylor imposed certain conditions which were later removed by Aris (1956)

Numerous authors utilised the Taylor's approach to discuss the dispersion in Newtonian and non-Newtonian fluids by considering different situations Fan and Hwang (1965) considered powerlaw fluid while Fan and Wang (1966) investigated the diffusion in Bingham and Ellis models Reiner-Phillippoff model was studied by Ghoshal (1971) Shah and Cox (1974) solved the problem for Eyring model using Taylor's approach

It may be also noted that, in a wide range of problems, the solute may have homogeneous and heterogeneous reactions with the solvent during the process of dispersion. For steady state conditions, some authors Katz (1959), Walker (1961), Solomon and Hudson (1967) considered the effects of homogeneous and heterogeneous reactions on dispersion Recently Gupta et al (1972) discussed the effect of homogeneous and heterogeneous reactions on the dispersion in the laminar flow of a Newtonian fluid between two plates

In all these investigations, the effect of viscosity variation across the tube has not been taken into consideration. However, for fluids of suspensions such as blood, the viscosity varies across the tube, Haynes et al (1959), Haynes (1960), Whitmore (1968), Middleman (1972) Also certain biological fluids such as blood, behave like non-Newtonian fluids under certain conditions, Charm and Kurland (1962,1965), Cokelet et al (1963), Merrill et al (1963,1965), Whitmore (1968)

Keeping the above in view, an attempt has been made in this thesis to study the effects of viscosity variation and the non-Newtonian

character of the fluid on the dispersion by taking into account the homogeneous reaction of the solute with the solvent

1 5 SUMMARY

This thesis consists of six chapters Chapter I gives the general introduction and a survey of the work related to blood flow through elastic arteries, motion of spermatozoa through the cervical canal and Taylor diffusion through non-Newtonian fluids

In Chapter II we have investigated the effect of elasticity and branching on flow characteristics of the blood by taking into account its non-Newtonian behaviour It has been pointed out that the behaviour of the blood which is non-Newtonian in smaller vessels, together with frequent branchings in the pre-arteriolar sections causes greater resistance to flow and are mainly responsible for a major pressure drop in this region

In Chapter III, a generalised Reynolds equation for the motion of spermatozoa in a channel in a two layer fluid with interaction of the wall is derived The interaction of the wall has been assumed to cause a force proportional to the shear-stress at the wall The following particular cases have been discussed in detail and the long wave length solution is obtained in each case, Shack and Lardner (1974)

(i) Motion of spermatozoa in a channel in the presence of wall interaction

(ii) Swimming of spermatozoa in a two layer fluid between parallel plates

It is found that the propulsive velocity of the sperm increases in the above cases

Chapter IV deals with the Taylor's dispersion of a solute with homogeneous and heterogeneous reactions in a fluid with varying viscosity flowing under laminar conditions in a channel or a tube The effects of viscosity variation on equivalent dispersion coefficient have been studied and it has been noted that it increases as the viscosity of the fluid decreases from the centre of the duct towards the wall

By using Taylor's approach in the last two chapters of the thesis the dispersion of matter through non-Newtonian fluids has been investigated by taking into consideration the effects of chemical reaction in the fluid The cases of Casson and Bingham fluid models have been considered It has been noted that the effective dispersion coefficient decreases due to non-Newtonian behaviour

REFERENCES

Atabek, H B (1968)
Wave propagation through a viscous fluid contained in a tethered, initially stressed, orthotropic elastic tube
Biophys J , Vol 8, p 626

Attinger, E O (1964)
Pulsatile blood flow
McGraw-Hill, New York

Aris, R (1956)
On the dispersion of a solute in a fluid flowing through a tube
Proc Roy Soc , Series A, Vol 235, p 67

Bergel (Ed) (1972)
Cardiovascular Fluid dynamics, Vols I and II
Academic Press

Brokaw, C J (1972)
Computer simulation of flagellar movement
Biophys J , Vol 12, p 584

Burton, A.C (1965)
Physiology and Biophysics of the circulation
Year Book Medical Publishers, Chicago

Cameron, A. (1966)
The principles of lubrication,
Wiley, New York

Casson, N (1958)
A flow equation for pigment oil suspensions of the printing ink type
In "Rheology of Disperse Systems"
(Ed) C C Mills, Pergamon Press, p 84

Charm, S E , and Kurland, G S (1962)
The flow behaviour and shear stress-shear rate characteristics of canine blood
Amer J Physiology, Vol 203, p 417

____________ (1965)
Viscometry of human blood for shear rates of 0-100,000 s^{-1}
Nature, Vol 206, p 617

Charm, S E and urland, G S (1967)
Static method for determining blood yield stress
Nature, Vol 216, p 1121

_______________ (1972)
Blood Rheology
In 'Cardiovascular fluid dynamics' (ed) Bergel, D H ,
Academic Press

Charm, S E , McComis,W and Kurland, G S (1964)
Rheology and structure of blood suspensions
J Appl Physiol , Vol 19,p 127

Chien, S , Usami, S , Dellenbeck, R J (1967)
Blood viscosity Influence of erythrocyte deformation
Science, Vol 157,p 827

Copley,A.L et al (1960)
Apparent viscosity and wall adherence of blood vessels
In 'Flow properties of blood and other biological systems'
(ed) A Copley and G Stainby, Pergamon Press,p 97

Cokelet et al (1963)
Rheology of human blood-measurement near and at zero
shear rate
Trans Soc Rheol , Vol 7, p 303

Cox, R H (1968)
Wave propagation through a Newtonian fluid contained in a
thick-walled visco-elastic tube
Biophys J , Vol 8, p 691

Crank, J (1975)
The mathematics of diffusion
Clarendon Press, Oxford

Crank, J and Gupta, R S (1972)
A moving boundary problem arising from the diffusion of
oxygen in absorbing tissue
J Inst Math and its appl , Vol 10, p 19

Crank, J and Park, G S (1968)
Diffusion in Polymers
Academic Press

Cumming G , Crank, J , and Parker, I (1966)
Gaseous diffusion in airways of human lung
Resp Physiol , Vol 1, p 58

Davajan et al (1970)
Spermatozoa transport in cervical mucus
Obstetric Gynec Survey, Vol 25, p 1

_______________ (1971)
A simplified technique for evaluation of biophysical properties of cervical mucus
Amer J Obstet Gynec , Vol 109, p 1042

Elstein et al (1971)
Ultra structure of cervical mucus
J Obstet Gynec Brit Comm , Vol 78, p 180

Fan, L T and Hwang, W S (1965)
Dispersion of Ostwald-de-Waele fluid in laminar flow through a cylindrical tube
Proc Roy Soc , Series A, Vol 283, p 576

Fan, L.T and Wang, C B (1966)
Dispersion of matter in non-Newtonian laminar flow through circular tube
Proc Roy Soc , Series A, Vol 292, p 203.

Ferguson, G G and Roach, M R (1972)
Flow conditions at bifurcations
In'Cardiovascular fluid dynamics' Vol II, (ed) Bergel, D.H ,
Academic Press, p 141

Fung, Y C (1968)
Biomechanics
Appl Mech Rev , Vol 21, No 1, p 1

Fung, Y C , Perrone, Y C and Anliker, M (1972)
Biomechanics Its foundations and objectives
Prentice Hall, N J

Ghoshal, S (1971)
Dispersion of solutes in non-Newtonian flows through a circular tube
Chem. Eng. Sc , Vol 26, p 185

Gray, J and Hancock, G J (1955)
The propulsion of Sea-Urchin spermatozoa
J Exp Biol , Vol 32, p 802

Gill, W N , and Sankara Subramanian, R (1970)
Exact Analysis of unsteady convective diffusion
Proc Roy Soc , Series A, Vol 316, p 341

——— (1971)
Dispersion of non-uniform slug in time dependent flow
Proc Roy Soc , Series A, Vol 322, p 101

Gupta, P S and Gupta, A S (1972)
Effect of homogeneous and heterogeneous reactions on the dispersion of a solute in the laminar flow between two plates
Proc Roy Soc , Series A, Vol 330, p 59

Hancock, G J (1953)
The self propulsion of microscopic organisms through liquids
Proc Roy Soc , Series A, Vol 217, p 96

Haynes, R H (1960)
Physical bases of the dependence of blood viscosity on tube radius
Am J Physiol , Vol 198, p 1193

Haynes, R H and Burton, A C (1959)
Role of non-Newtonian behaviour in hemodynamics
Am J. Physiology, Vol 197, p 943

Hershey, D and Cho, S J (1966)
Laminar flow of suspensions (blood) Thickness and effective slip velocity of the film adjacent to the wall
In "Chemical Engineering in Medicine", Chem Engg Progress Symposium Series 62, (ed) Leonard, E F , A I Ch E., Philadelphia, p 139

Huckaba et al (1968)
A generalised approach to the modelling of arterial blood flow
Bull Math Biophys , Vol 30, p 645

Katz, S (1959)
Chemical reactions catalysed on a tube wall
Chem Engng Sc , Vol 10, p 202

Lightfoot, E N (1974)
Transport phenomena and living systems
Wiley Interscience, New York

Lighthill, M J (1974)
Mathematical Bio-fluid Dynamics
S I A M

Lighthill, M J (1975)
Mathematics of aquatic animal locomotion at low Reynolds numbers
Winter School in Physiological fluid dynamics, I I T , Delhi, Dec 8 - 18, 1975

McDonald, D A (1974)
Blood flow in arteries
Baltimore Williams and Wilkins Co

Merrill et al (1963)
Non-Newtonian Rheology of human blood Effect of fibrinogen deduced by subtraction
Circul Res , Vol 13, p 48

__________ (1964)
Rheology of human blood and red cell plasma membrane
Bibli Anat , Vol 4, p 51

__________ (1965)
The casson equation and rheology of blood near shear zero
In "Proceedings of fourth international congress on rheology", part 4, (Ed) Copley, A.L , Interscience Publishers, p 135

Moghissi, K S (1969)
Sperm migration in human female genital tract
J Reprod Med , Vol 3, p 73

__________ (1971)
Sperm migration through cervical mucus
In "Pathways to conception" (Ed) A Sherman, Thomas, Springfield

Moghissi, K S and Blandau, R J (1972)
Biology of the cervix. Summary of a symposium held at Lake Wilderness, Washington, June 9 - 12, 1971
J Reprod Med , Vol. 8, p 21

Middleman, S (1972)
Transport phenomena in the cardiovascular system
Wiley Interscience, New York

Mirsky, I (1967)
Wave propagation in a viscous fluid contained in an orthotropic elastic tube
Biophys J , Vol 7, p 165

Odeblad, E (1959)
The physics of the cervical mucus
Acta Obstet Gynec Scand , Vol 38, Suppl 1, p 44

__________ (1962)
Undulations of macromolecules in cervical mucus
Int J Fert , Vol 7, p 313

__________ (1968)
The functional structure of cervical mucus
Acta Obstet Gynec Scand , Vol 47, Suppl 1, p 59

__________ (1971)
Cervical factors Nobel symposium 15, Control of Human Fertility, (Ed) Diczfalusy, E and Borell, U , Wiley, p 89

Pironneau, O (1975)
The fluid mechanics of uniflagellated micro-organisms
Winter School in Physiological Fluid Dynamics, I I T Delhi, Dec 8 - 18, 1975

Rodkiewicz, C M and Howell, D H. (1971)
Fluid dynamics in a large arterial bifurcation
AIAA J , Vol 9, p 2284

Rodkiewicz, C M and Roussel (1973)
Fluid Mechanics in a large Arterial bifurcation
Trans. ASME, March 1973, p 108

Reynolds, A.J (1965)
The swimming of minute organisms
J. Fluid Mech., Vol. 23, p 241

Reiner and Scott-Blair (1959)
The flow of blood through narrow tubes
Nature, Vol 184, p 354

Shack, W J and Lardner, T J (1974)
A long wave length solution for a micro-organism swimming in a chennel
Bull Math. Biol , Vol 36, p 435

Shah, S N and Cox, K E (1974)
Dispersion of solutes in non-Newtonian laminar flow through a circular tube Eyring model fluid
Chem Engg Sc , Vol 29, p 1282

Skalak, R (1972)
Synthesis of complete circulation
In "Cardiovascular fluid dynamics" (Ed) Bergel, D H , Academic Press, p 341

Sobrero, A.J (1963)
Sperm migration in the female genital tract
In "Mechanisms Concerned with Conception" (ed) Hartman C G
Macmillan, London p 173

Solomon, R L and Hudson, J L (1967)
Heterogeneous and homogeneous reactions in a tubular reactor
A.I Ch E J , Vol 13, p 545

Smelser, R (1972)
A low Reynolds number flow problem with application to spermatozoan transport in cervical mucus
M.S thesis, Massachusetts Institute of Technology, Cambridge

Smelser, R et al (1974)
The swimming of spermatozoa in an active channel
J Biomech , Vol 7, No 4, p 349

Taylor, Sir Geoffrey (1951)
Analysis of Microscopic organisms
Proc Roy Soc , Series A, Vol 209, p 447

__________ (1953)
Dispersion of soluble matter in solvent flowing slowly through a tube
Proc Roy Soc Series A, Vol 219, p 186

__________ (1954)
Dispersion of matter in turbulent flow through a pipe
Proc Roy Soc , Series A, Vol 223, p 446

__________ (1954)
Conditions under which dispersion of a solute in a stream of solvent can be used to measure molecular diffusion
Proc Roy Soc , Series A, Vol 225, p 473

Walker, P (1961)
Chemical reaction and diffusion in a catelytic tubular reactor
Phys Fluids, Vol 4, p 1211

Whirlow, D K and Rouleau, W T (1965)
Periodic flow of a viscous liquid in a thick-walled elastic tube
Bull Math Biophys , Vol 27, p 355

Whitmore, R.L (1968)
Rheology of circulation
Pergamon Press

Womersley, J P (1957)
An elastic tube theory of pulse transmission and oscillatory flow in mammalian arteries
Wright Air Development Centre, WADC Report TR 56-614

______________ (1958)
Oscillatory flow in arteries The Constrained elastic tube as a model of arterial flow in pulse transmission
Phys Med Biol , Vol 2, p 178

Wooding, R A (1960)
Instability of viscous liquid of variable density in a vertical Hele-Shaw cell
J Fluid Mech , Vol 7, p 510

CHAPTER II

NON-NEWTONIAN BEHAVIOUR OF BLOOD THROUGH ELASTIC ARTERIES

2.1 INTRODUCTION

It is well known that the walls of blood vessels are elastic and distensible. Elastin fibre which is present in the tissues of the walls is mainly responsible for their elastic nature. To understand the realistic behaviour of the blood flow in arteries, this effect should be taken into account. Euler (1775) was the first to study the wave transmission in an elastic pipe by considering the blood as an incompressible inviscid fluid. Later Womersley (1957) discussed a theory of pulse transmission and oscillatory flow in mammalian elastic arteries. Mirsky (1967) studied the effects of anisotropy by taking into consideration the longitudinal tethering of the wall while Cox (1968) considered the effects of viscoelasticity. The pulsatile flow in an elastic tube has been discussed by Middleman (1972), Lightfoot (1974), McDonald (1974). The effects of branching also have been taken into consideration, Lightfoot (1974).

In these studies the blood has been considered as a Newtonian fluid. However the blood is a particulate suspension of red cells in plasma and it behaves like a non-Newtonian fluid, Merrill et al (1963), Hershey and Cho (1966), Huckaba et al (1968), Charm and

Kurland (1962, 1965), Whitmore (1968), Bergel (1972) It has been pointed out by Charm and Kurland (1965), Huckaba et al (1968), Hershey and Cho (1966) that the blood may be characterised as a powerlaw fluid, when the shear rate is less than 20 sec^{-1} However, Reiner and Scott-Blair (1959) suggested that the blood behaves as a Casson model fluid. It has been shown that Casson model fits well, for shear rates, 1-100,000 sec^{-1}, Charm and Kurland (1962, 1965), 0 1-20 sec^{-1}, Merrill et al (1963), for a wide range of haemotocrit values

It may be noted that the cardiovascular system is a very complex net work and the behaviour of the blood flow through it depends, in general, upon the nature of the blood vessels such as elasticity, tapering, curvature, branchings etc , the non-Newtonian behaviour of the blood, the nature of the flow such as pulsatile, laminar, etc It is not mathematically tractable to have a complete realistic model with all these complexities and "some degree of approximation and restriction of the generality of the model must be adopted to make any progress at all", Bergel (1972)

To study the performance of any vascular bed, the concept of peripheral resistance or resistance to flow, which may be defined as the ratio of the pressure drop to the flux is most useful, Burton (1965), Bergel (1972) In general the peripheral resistance (PR) depends upon the geometry of the bed, rheology of the blood and the flux Q etc In a complete vascular bed the total resistance to flow is the algebraic

sum of the resistances in arteries, arterioles, capillaries, veins, connected in series. From the definition of peripheral resistance it can be seen that for a given flux, the PR is proportional to the pressure drop. The resistance in the proximity of arterioles constitutes a major share of the total resistance. The mean arterial pressure of 100 mm Hg drops to 30-35 mm Hg in arterioles, McDonald (1974).

Keeping the above in view, in this chapter, we investigate the effects of elasticity and branching on the peripheral resistance in the arteriolar and pre-arteriolar section of the vascular bed by characterising the blood as Casson and powerlaw fluids.

2.2 BASIC EQUATIONS

Consider an axisymmetrical flow of a non-Newtonian fluid in a circular elastic artery whose physical configuration and the corresponding coordinate system are shown in the figure (2.1).

The Navier-Stokes equations governing the flow in this case are

$$\rho \left[v \frac{\partial v}{\partial r} + u \frac{\partial v}{\partial z} \right] = -\frac{\partial p}{\partial r} - \left[\frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rr}) - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z} \right] \tag{2.1a}$$

$$\rho \left[v \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial z} \right] = -\frac{\partial p}{\partial z} - \left[\frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z} \right] \tag{2.1b}$$

The continuity equation can be written as

$$\frac{1}{r} \frac{\partial}{\partial r} (rv) + \frac{\partial u}{\partial z} = 0 \tag{2.2}$$

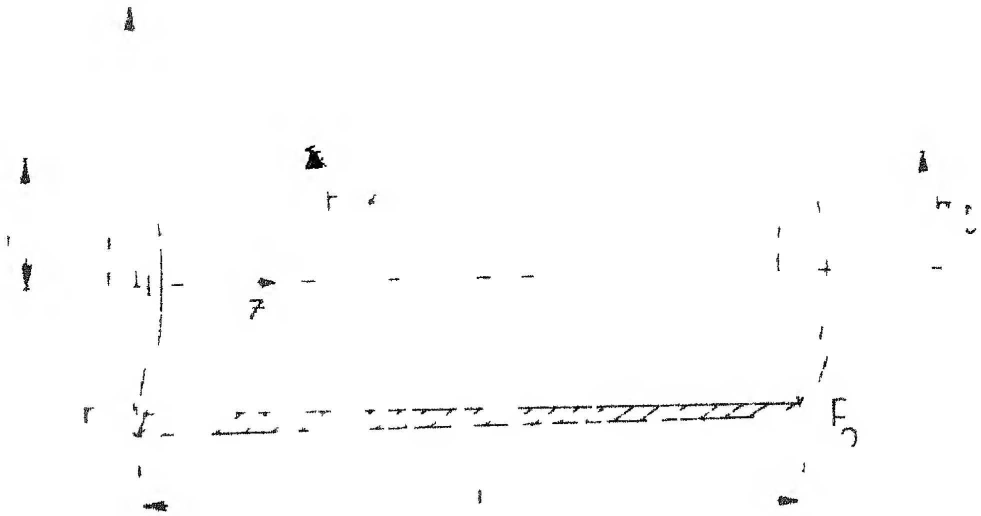

FIG 21 ELASTIC ARTERY

To simplify the model we assume that the displacement of the wall is small so that the radial component v of the velocity is negligible when compared to the axial component u, Skalak in Fung (1966) Since the Reynolds number of the flow is small, [for flows in the region of terminal branches Re = 9 and near arterioles it is 0 03 , Lightfoot (1974)], the inertia terms are neglected, Batchelor (1967)

With these assumptions, the equations of motion reduce to

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tau) = -\frac{\partial p}{\partial z} \tag{2.3}$$

$$0 = -\frac{\partial p}{\partial r} \tag{2 4}$$

The shear stress at any radial distance r is given by, Bird et al (1960)

$$\tau = -\frac{r}{2}\frac{dp}{dz} \tag{2 5}$$

If τ_R is the wall shear, this gives

$$\tau_R = -\frac{R}{2}\frac{dp}{dz} \tag{2.6}$$

On using equations (2 5) and (2 6) we get

$$\frac{\tau}{\tau_R} = \frac{r}{R} \tag{2 7}$$

The constitutive equation for a non-Newtonian fluid, in general, can be given as

$$- \frac{du}{dr} = f(\tau) \tag{2 8}$$

The flux Q is defined as

$$Q = 2\pi \int_0^R r\,u\,dr = -\pi \int_0^R r^2 \frac{du}{dr}\,dr \tag{2 9}$$

Using the equation (2 7) and (2 8) we get

$$Q = \frac{\pi R^3}{\tau_R^3} \int_0^{\tau_R} \tau^2 \quad f(\tau)\,d\tau \tag{2 10}$$

The peripheral resistance (PR), denoted by λ, is defined as

$$\lambda = \frac{\Delta p}{Q} \tag{2 11}$$

where Δp is the pressure drop across the bed considered

It can be seen from equation (2 11) that, for a given flux, the PR is proportional to the pressure drop, McDonald (1974)

2 3 CASSON MODEL FLUID

The stress-strain law for Casson model fluid is given by

$$\tau^{1/2} = \eta\left(-\frac{du}{dr}\right)^{1/2} + \tau_o^{1/2} \tag{2 12}$$

where τ_o and η are the yield stress and consistency Hence

$$f(\tau) = \frac{1}{\eta^2}\,[\tau + \tau_o - 2\sqrt{\tau\tau_o}\,] \tag{2 13}$$

Using equation (2 13) in (2 10) we get

$$Q = \frac{\pi R^3}{4\eta^2}\tau_R\,[1 + \frac{4}{3}\left(\frac{\tau_o}{\tau_R}\right) - \frac{16}{7}\left(\frac{\tau_o}{\tau_R}\right)^{1/2} - \frac{1}{21}\left(\frac{\tau_o}{\tau_R}\right)^4\,] \tag{2 14}$$

This expression is also given in Whitmore (1968) Approximating for small (τ_o/τ_R) we get

$$Q = \frac{\pi R^3}{4\eta^2}\ \tau_R\ [1 - \frac{8}{7}\sqrt{(\tau_o/\tau_R)}]^2 \tag{2 15}$$

Eliminating τ_R from equations (2 6) and (2 15) we obtain

$$\frac{dp}{dz} = \frac{-2}{R}\ [\ \frac{8}{7}\sqrt{\tau_o} + 2\eta\sqrt{Q/\pi R^3}\,]^2 \tag{2 16}$$

where R , the radius at any axial distance z , depends upon the elastic nature of the artery.

The relation between the pressure and the stretched radius $R(z)$ of the tube can be written as, Rashevsky (1945), Morgan (1952),

$$p = Eh[\frac{1}{R_o} - \frac{1}{R}] \tag{2 17}$$

where E is the Young's modulus of elasticity, R_o is the unstretched radius of the tube and h is the wall thicknes.

Differentiating the equation (2 17) with respect to z we get

$$\frac{dp}{dz} = \frac{Eh}{R^2}\frac{dR}{dz} \tag{2.18}$$

Eliminating $(\frac{dp}{dz})$ from equations (2 16) and (2 18), we get

$$\frac{dR}{dz} = -[\,a\,R^{1/2} + \frac{b}{R}]^2 \tag{2.19}$$

where

$$a = \frac{8}{7}\sqrt{2\tau_o/Eh} \tag{2.20}$$

$$b = 2\eta\sqrt{2Q/Eh\pi} \tag{2 21}$$

Though the equation (2 19) can be integrated it is difficult to get R as an explicit function of z To achieve this, we solve this equation by iteration with

$$\tau_0 = 0 \text{ , } R = R_1(z)$$

as starting values, where $R_1(z)$ satisfies the following equation

$$\frac{dR_1}{dz} = \frac{-b^2}{R_1^2} \tag{2 22}$$

The initial condition for this equation is

$$z = 0, \quad R_1 = R_i \tag{2 23}$$

where R_i, the radius at the entrance , is given by [from equation (2.17)]

$$R_i = R_0 / [1 - p_i R_0 / Eh] \tag{2 24}$$

Now, integrating equation (2 22) and using (2.23), we have

$$R_1 = R_i (1 - \frac{\delta z}{\ell})^{1/3} \tag{2 25}$$

where

$$\delta = \frac{24 \ \eta^2 \ Q \ell}{Eh \ \ \pi R_i^3} \tag{2 26}$$

Iterating the equation (2 19), we have

$$\frac{dR_2}{dz} = -[a R_1^{1/2} + \frac{b}{R_1}]^2 \tag{2 27}$$

where R_2 is the first iterated value of $F(z)$

Integrating the equation (2 27) using the initial condition

$$z = 0 \ , \ R_2 = R_1 \qquad (2\ 28)$$

we have

$$R_2 = R_i \left(1 - \frac{\delta z}{\ell}\right)^{1/3} + \frac{4}{49}\left(\frac{\tau_o \qquad \pi R_1^4}{\eta^2 \ Q}\right)\left[\left(1 - \frac{\delta z}{\ell}\right)^{4/3} - 1\right]$$

$$+ \frac{16}{35}\frac{R_1^{5/2}}{\eta}\left(\frac{\tau_o \quad \pi}{Q}\right)^{1/2}\left[\left(1 - \frac{\delta z}{\ell}\right)^{5/6} - 1\right] \qquad (2\ 29)$$

For $\delta << 1$ this can be approximated as

$$R_2 = R_1 \ \left[1 - \frac{\phi z}{\ell}\right] \qquad (2\ 30)$$

where

$$\phi = \frac{2\ell}{Eh}\left[\frac{8}{7} \ \sqrt{\tau_o} + 2n\sqrt{\left(\frac{Q}{\pi R_1^3}\right)}\right]^2 \qquad (2\ 31)$$

If R_ℓ denotes the radius at $z = \ell$, we have from equation (2 30)

$$R_\ell = R_1 \ \left[1 - \phi\right] \qquad (2.32)$$

From equation (2.17), we obtain

$$p_1 - p_o = Eh\left[\frac{1}{R_\ell} - \frac{1}{R_i}\right] \qquad (2\ 33)$$

(where p_o is the pressure at $z = \ell$) which, on using equation (2 32) can be written as

$$p_1 - p_o = \frac{Eh\ \phi}{(1-\phi)\ R_i} \qquad (2\ 34)$$

The peripheral resistance, denoted by λ, is given by, [see equation (2 11)]

$$\lambda = \frac{2\ell}{Q R_1} \left[\frac{\left(\frac{8}{7}\sqrt{\tau_o} + 2\eta\sqrt{Q/\pi R_1^3}\right)^2}{1 - \frac{2\ell}{Eh}\left(\frac{8}{7}\sqrt{\tau_o} + 2\eta\sqrt{Q/\pi R_1^3}\right)^2} \right] \tag{2 35}$$

In the case of Newtonian fluid ($\tau_o = 0$) flowing in an elastic pipe, the peripheral resistance λ_N, for the same flux Q, is given by

$$\lambda_N = \frac{8\,\eta^2\,\ell}{\pi R_1^4}\left[1 - \frac{8\eta^2\,Q\,\ell}{Eh\,\pi\,R_1^3}\right]^{-1} \tag{2 36}$$

The corresponding peripheral resistance for a Casson model fluid flowing in a rigid pipe is given by (as $E \to \infty$)

$$\lambda^* = \frac{2\ell}{R_o}\left[\frac{8}{7}\sqrt{\left(\frac{\tau_o}{Q}\right)} + 2\eta\sqrt{1/\pi R_o^3}\,\right]^2 \tag{2 37}$$

For a Newtonian fluid flowing in a rigid tube, we get

$$\lambda_N^* = \frac{8\eta^2\ell}{\pi R_o^4} \tag{2 38}$$

a well known particular case

Dividing the equation (2 35) by equation (2 36) we get

$$\frac{\lambda}{\lambda_N} = \left[\frac{\left\{1 + \frac{4}{7\eta}\sqrt{\left(\frac{\tau_o\,\pi\,R_i^3}{Q}\right)}\right\}^2 - \frac{2\ell}{Eh}\left(\frac{8}{7}\sqrt{\tau_o} + 2\eta\sqrt{Q/\pi R_1^3}\right)^2}{1 - \frac{2\ell}{Eh}\left(\frac{8}{7}\sqrt{\tau_o} + 2\eta\sqrt{Q/\pi R_i^3}\right)^2} \right] \tag{2 39}$$

When $E \to \infty$, we get the ratio in the case of rigid tube as

$$\frac{\lambda^*}{\lambda_N^*} = [1 + \frac{4}{7\eta}\sqrt{\frac{\tau_o \ \pi R_o^3}{Q}}]^2 \qquad (2.40)$$

It can be seen that the numerator of the term in the square bracket of equation (2 39) is greater than the denominator, which implies that

$$\lambda > \lambda_N$$

Hence the PR is greater in the case of Casson model fluid than the Newtonian model. The same result holds in the case of rigid tube also, as seen from equation (2 40) For ϕ <<1, we get the PR for Casson model fluid flowing in an elastic artery (neglecting ϕ^2 and higher powers) as

$$\lambda = \frac{2\ell}{R_1}[\frac{8}{7}\sqrt{\frac{\tau_o}{Q}} + 2\eta\sqrt{\frac{1}{\pi R_1^3}}]^2 \qquad (2.41)$$

It may be noted here that as $E \to \infty$, R_1 decreases to R_o and hence λ increases to λ^*. Thus PR in the case of elastic tube is less when compared to the case of rigid tube for the same flux.

From the above analysis it may be noted that PR increases as the elasticity of the wall decreases and it is enhanced further by the non-Newtonian behaviour of the blood Thus, the non-elastic nature of blood vessels and non-Newtonian behaviour of the blood are responsible for a major pressure drop occuring in the arteriolar section of the vascular bed which is comparitively rigid, McDonald (1974). See Figs (2 2) to (2 4)

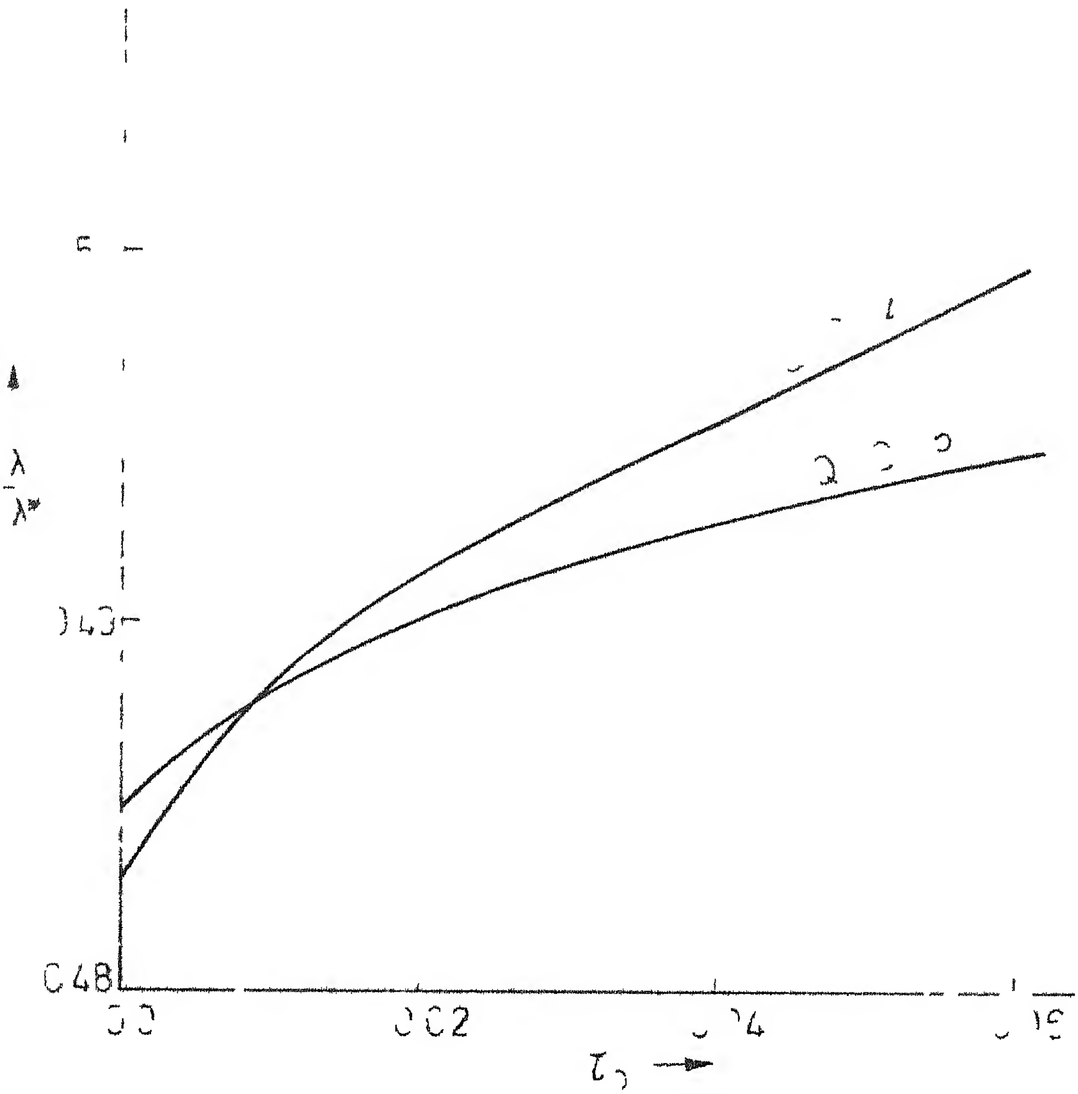

FIG 2 2 RATIO OF FR IN ELASTIC AND RIGID TUBES E = 6x10 dyne cm, p... dyne cm^{-2} η^2 = 035 dyne cm^{-2}/sec R = 03 cm

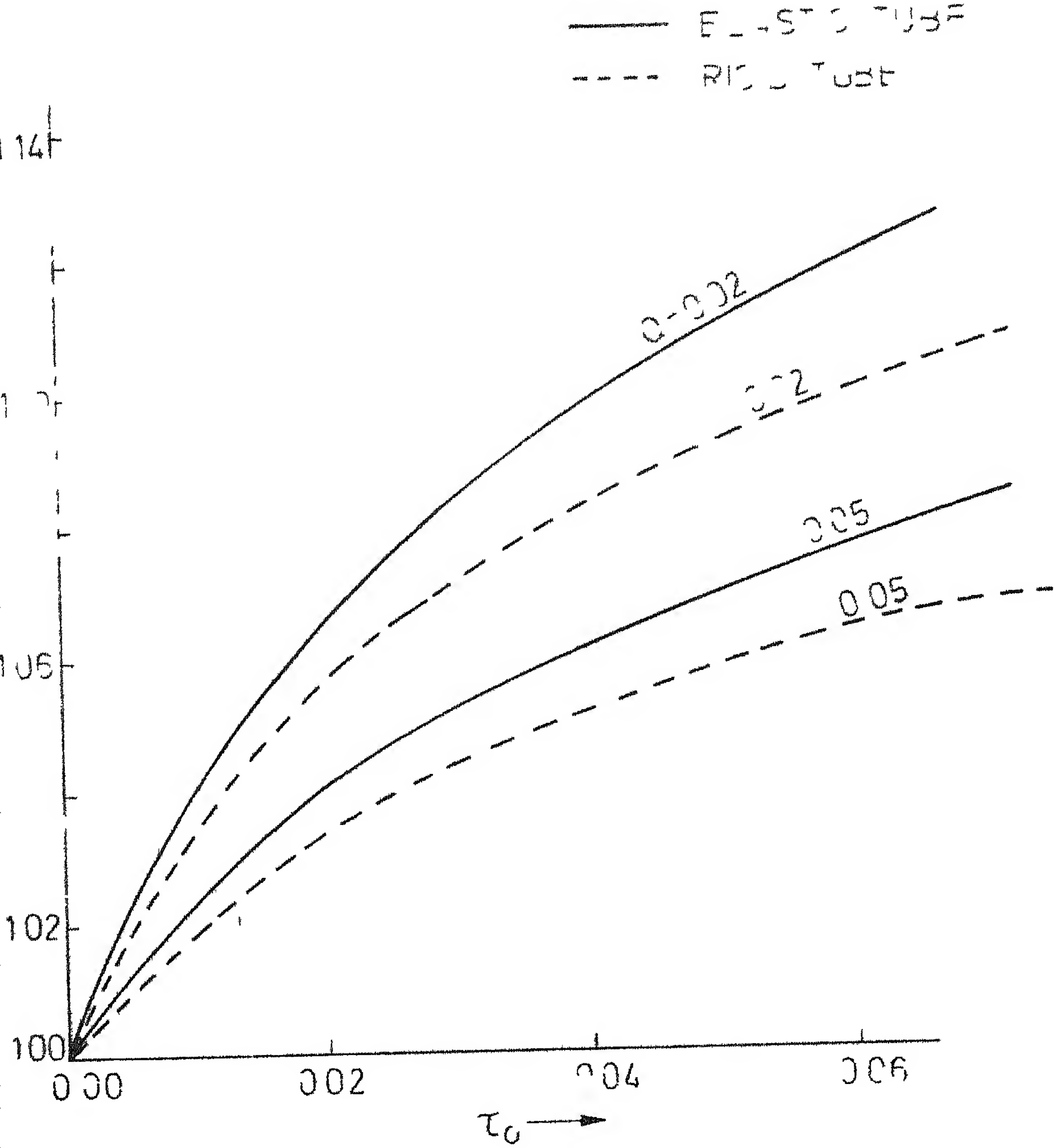

FIG 2.3 RATIO OF PR OF CASSON AND NEWTONIAN FLUIDS $E = 6 \times 10^6$ dyne cm^{-2}, $p_1 = 10^5$ dyne cm^{-2} $\eta^2 = 0.035$ dyne cm^{-2} sec, $R_0 = 0.3$ cm

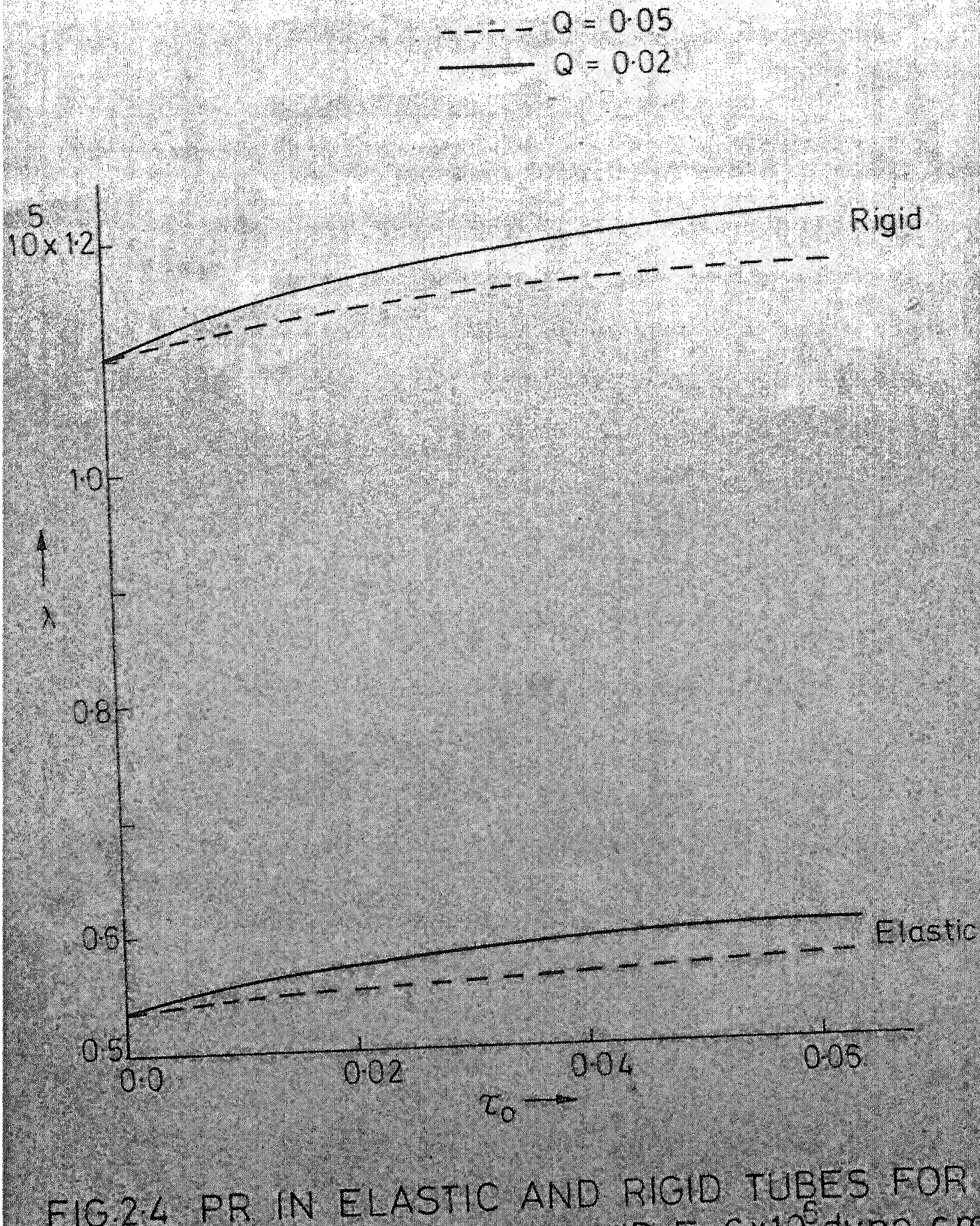

FIG.2·4 P.R. IN ELASTIC AND RIGID TUBES FOR CASSON MODEL FLUID, $E = 6\times10^{6}$ dyne cm^{2} $p_i = 10^{5}$ dyne cm^{-2}, $\eta^{2} = \cdot035$ dyne cm^{-2} sec, $R_0 = 0{\cdot}03$ cm

In larger vessels, walls are more elastic and blood behaves like a Newtonian fluid causing less FR and less pressure drop

2.4 EFFECTS OF BRANCHINGS IN THE VASCULAR BED

In this section we study the effects of non-elastic branchings in an elastic tube by considering the blood as a Casson model fluid. The physical configuration, which is more or less close to the natural situation in the arteriolar and pre-arteriolar section of the vascular bed, is shown in Fig. (2.5). Here we assume that the main artery of length $\beta\ell\,(0 < \beta < 1)$ diverges into M branches at the point B. The main artery is assumed to be elastic and the branches to be rigid, as the distensibility of arterioles is small when compared to that of the main tube, Burton (1965), Lightfoot (1974). The radius of the main tube is denoted by $R(z)$ and that of jth branch by R_j. The flux in the main tube is denoted by Q and the flux in the jth branch by Q_j. It is further assumed that all branches are of the same length $(1-\beta)\ell$ and the pressure at the end of each branch is p_o. We also neglect bending, tapering and entrance effects, Bergel (1972), Krovetz (1965).

If p_1 denotes the pressure at the inlet and p_b denotes the pressure at the branching point, then the pressure drop in the main tube can be written from equation (2.41), as follows

$$p_1 - p_b = \frac{2\beta\ell}{R_1}\left[\frac{8}{7}\sqrt{\tau_o} + 2\eta\sqrt{Q/\pi R_1^3}\,\right]^2 \qquad (2.42)$$

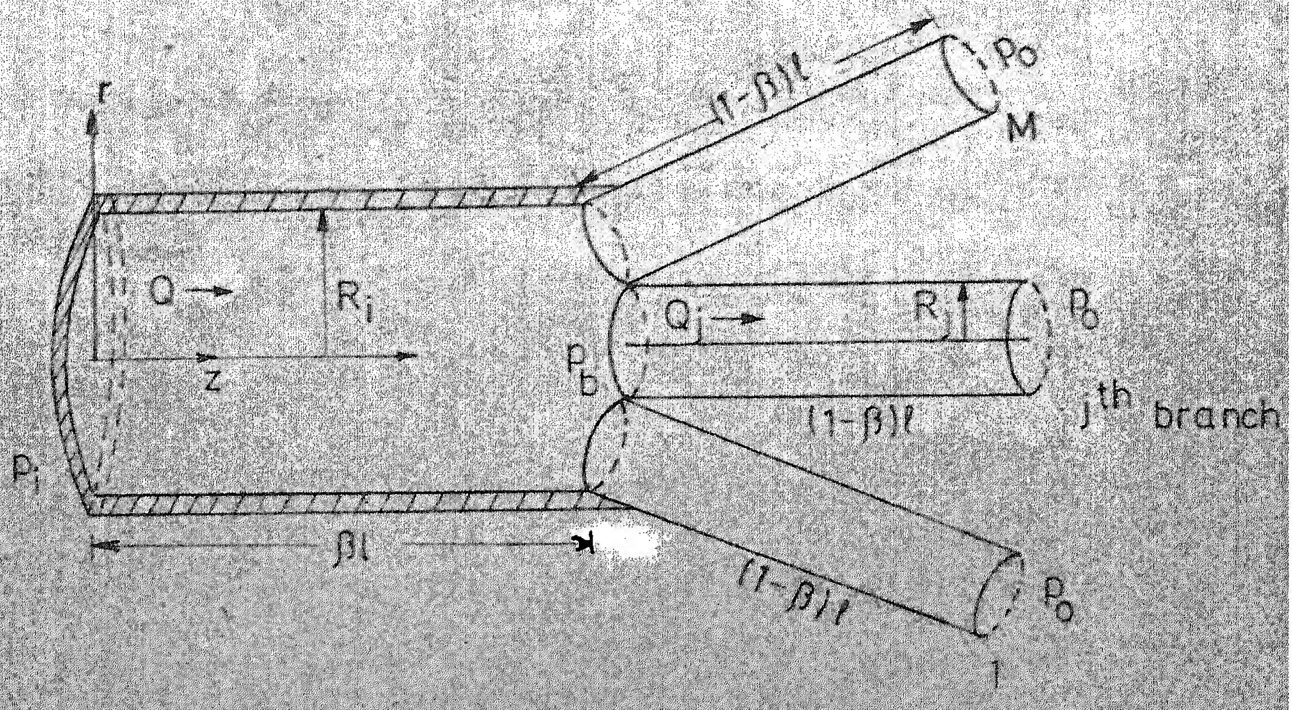

FIG.2.5 A BRANCHED VASCULAR BED

where R_1 is the elongated radius of the main tube. The pressure drop in the jth branch can be given by

$$p_b - p_o = \frac{2(1-\beta)\ell}{R_j}\left[\frac{8}{7}\sqrt{\tau_o} + 2\eta\sqrt{Q_j/\pi R_j^3}\right]^2 \qquad (2.43)$$

From equations (2.42) and (2.43) we have the expressions for the fluxes as

$$Q = \frac{\pi R_1^4}{8\eta^2}\left[\frac{128}{49}\frac{\tau_o}{R_1} + \frac{p_1-p_b}{\beta\ell} - \frac{16}{7}\sqrt{\frac{2\tau_o}{R_1}}\left(\frac{p_1-p_b}{\beta\ell}\right)^{1/2}\right] \qquad (2.44)$$

$$Q_j = \frac{\pi R_j^4}{8\eta^2}\left[\frac{128}{49}\frac{\tau_o}{R_j} + \frac{p_b-p_o}{(1-\beta)\ell} - \frac{16}{7}\sqrt{\frac{2\tau_o}{R_j}}\left\{\frac{p_b-p_o}{(1-\beta)\ell}\right\}^{1/2}\right] \qquad (2.45)$$

Using the condition of continuity, we have

$$Q = \sum_{j=1}^{M} Q_j \qquad (2.46)$$

Substituting the expressions for Q and Q_j from equations (2.44) and (2.45) in equation (2.46), we can obtain the pressure p_b at the branching point from the following equation

$$p_b\left[1-\beta+\beta\sum_1^M \alpha_j^4\right] = \frac{128}{49}\frac{\tau_o\ell}{R_1}\beta(1-\beta)\left(1-\sum_1^M \alpha_j^3\right)$$

$$+ p_1(1-\beta) + p_o\beta\sum_1^M \alpha_j^4$$

$$-\frac{16}{7}\sqrt{\frac{2\tau_o\ell}{R_1}}\,\beta(1-\beta)\left[\sqrt{\left(\frac{G}{\beta}\right)} - \sqrt{\frac{p_1-p_o-G}{1-\beta}}\sum_1^M \alpha_j^{7/2}\right] \qquad (2.47)$$

where

$$\alpha_j = R_j/R_1$$

$$G = \frac{2\beta\ell}{R_1}\left[\frac{8}{7}\sqrt{\tau_o} + 2\eta\sqrt{Q/\pi R_1^3}\right]^2 \qquad (2.48)$$

Substituting the value of p_b in equation (2 42), the pressure drop in the vascular bed is given by the equation

$$\frac{p_1 - p_0 - G}{1-\beta} = \frac{G}{\beta \sum_1^M \alpha_j^4} + \frac{128}{49}\frac{\tau_o \ell}{R_1}\frac{(1-\beta)\left(1 - \sum_1^M \alpha_j^4\right)}{\sum_1^M \alpha_j^4}$$

$$- \frac{16}{7}\sqrt{\left(\frac{2\tau_o \ell}{R_1}\right)}\frac{(1-\beta)}{\sum_1^M \alpha_j^4}\left[\sqrt{\frac{G}{\beta}} - \sqrt{\left(\frac{p_1 - p_0 - G}{1-\beta}\right)}\sum_1^M \alpha_j^{7/2}\right] \quad (2\ 49)$$

which can be easily solved as a quadratic in $\left(\frac{p_1 - p_0 - G}{1-\beta}\right)^{1/2}$

To see the effects of various parameters, it is further assumed that all the branches are of equal radius i.e ,

$$\alpha = \alpha_1 = \alpha_2 = \qquad = \alpha_N \quad (2\ 50)$$

and then

$$Q_j = Q/M \quad (2\ 51)$$

In such a case, using (2 50) and (2 51) in the equation (2.49) and solving for $p_1 - p_0$, we get

$$p_1 - p_0 = \frac{(1-\beta)\,2\ell}{R_1}\left[\frac{8}{7}\sqrt{\frac{\tau_o}{\alpha}} + \frac{2\eta}{\alpha^2\sqrt{M}}\sqrt{\frac{Q}{\pi R_1^3}}\right]^2$$

$$+ \frac{\beta\, 2\ell}{R_1}\left[\frac{8}{7}\sqrt{\tau_o} + 2\eta\sqrt{Q/\pi R_1^3}\right]^2 \quad (2\ 52)$$

We notice that the above equation can also be obtained by adding the equations (2 42) and (2 43) and using (2 50) and (2 51)

It may be noted here that M and α are connected by the relation

$$d = M\alpha^2 \tag{2 53}$$

where d is the ratio of total area of cross section of the branches and the area of cross section of the main tube The average value of d is 1 28, McDonald (1974) and it ranges over 0 75, 1.02, 1 29 in different parts of the body, Caro et al (1971)

From equations (2.52) and (2 53), the peripheral resistance λ_B, in this case, can be written as

$$\lambda_B = \frac{2(1-\beta)\ell}{R_1}\left[\frac{8}{7}\sqrt{\left(\frac{\tau_o}{Q\alpha}\right)} + \frac{2n}{d}\sqrt{\left(\frac{M}{\pi R_1^3}\right)}\right]^2$$

$$+ \frac{2\beta\ell}{R_1}\left[\frac{8}{7}\sqrt{\left(\frac{\tau_o}{Q}\right)} + 2n\sqrt{\left(\frac{1}{\pi R_1^3}\right)}\right]^2 \tag{2 54}$$

When the main tube is considered to be rigid, then the corresponding PR is given by

$$\lambda_B^* = \frac{2(1-\beta)\ell}{R_o}\left[\frac{8}{7}\sqrt{\left(\frac{\tau_o}{Q\alpha^*}\right)} + \frac{2n}{d^*}\sqrt{\left(\frac{\bar{M}}{\pi R_o^3}\right)}\right]^2$$

$$+ \frac{2\beta\ell}{R_o}\left[\frac{8}{7}\sqrt{(\tau_o/Q)} + 2n\sqrt{(1/\pi R_o^3)}\right]^2 \tag{2 55}$$

where $\alpha^* = R_j/R_o$ and M, which is the number of branches, can take only integer values. Its minimum value is 2. It can easily be seen that, for the range of d stated above,

$$\frac{\sqrt{M}}{d} > 1 \tag{2 56}$$

Equation (2 56) and the fact $\alpha < 1$ enable us to conclude that

$$\lambda_B > \lambda \tag{2 57}$$

Hence the PR of the vascular bed increases in the presence of branchings

From equation (2 54) it may be further noted that the PR increases as the distensibility of the main tube decreases and the number of branches increases and it is enhanced by the non-Newtonian behaviour of the blood (See Fig (2 6))

It is interesting to compare the pressure gradients in the main and branch portions of the bed Dividing the equation (2 43) by equation (2 42), using (2 50), (2 51) and (2 53), we get the ratio of pressure gradients in the case when all the branches are of equal radius, as

$$\frac{P_2}{P_1} = \left[\frac{\frac{8}{7}\sqrt{\tau_o/Q\alpha} + \frac{2n}{d}\sqrt{\frac{M}{\pi R_1^3}}}{\frac{8}{7}\sqrt{\tau_o/Q} + 2n\sqrt{1/\pi R_1^3}} \right]^2 \tag{2 58}$$

where P_1 and P_2 are pressure gradients in the main and branched portion respectively

It can be seen that the numerator of the term on the right hand side is greater than the denominator Hence

$$P_2 > P_1$$

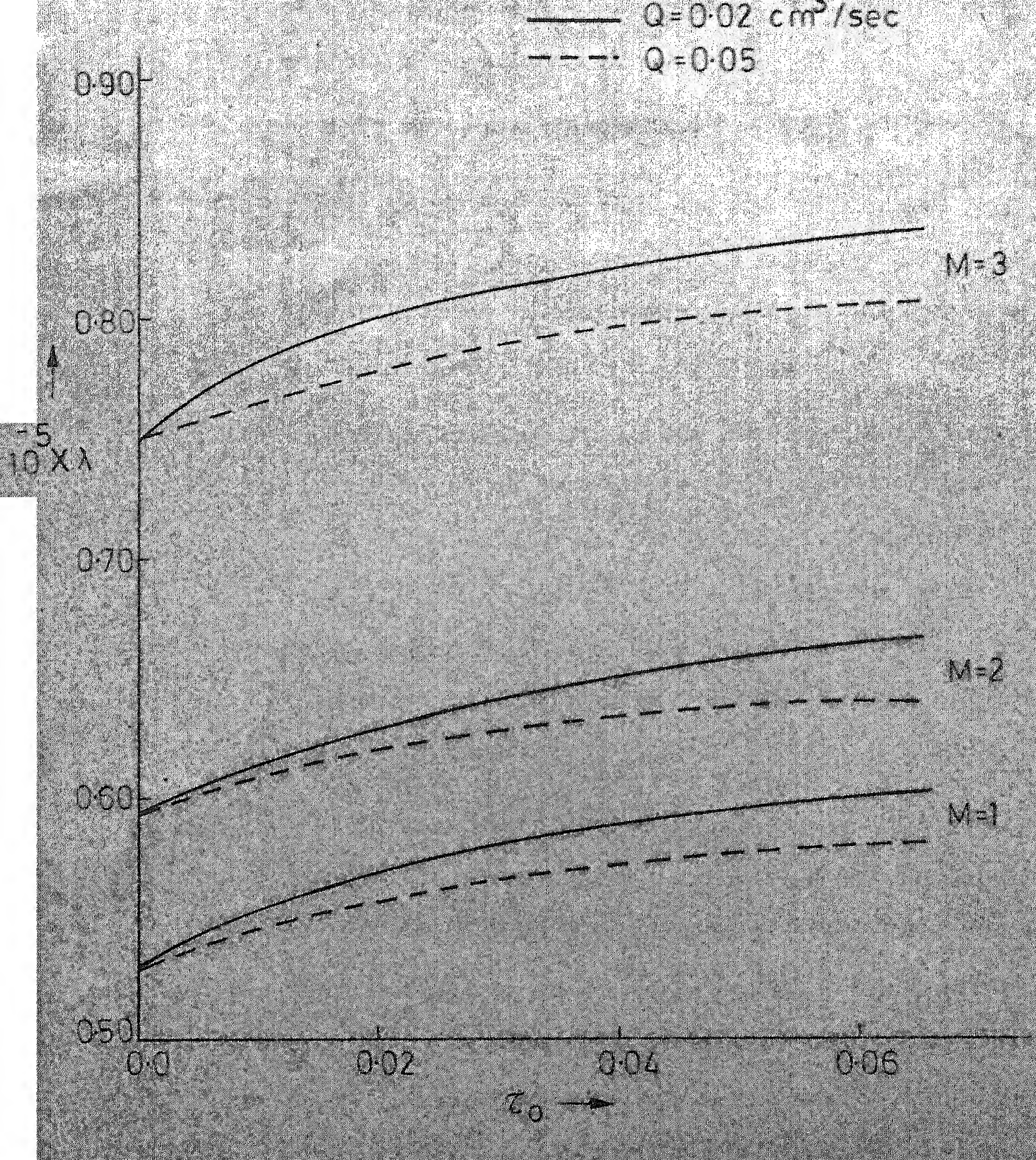

FIG 2·6 PERIPHERAL RESISTANCE IN A BRANCHED VASCULAR BED $E = 6 \times 10^6$ dyne cm^{-2}, $p_i = 10^5$ dyne cm^{-2} $\eta = \cdot 035$ dyne cm^{-2} sec, $R_0 = \cdot 03$ cm, $d = 1{\cdot}28$, $\beta = \cdot 5$

which shows that the pressure gradient in the branch is greater than that in the main tube A similar result for Newtonian fluid was obtained by McDonald (1974)

It may be seen further that for $\beta \leq \frac{1}{2}$, we get

$$\frac{p_b - p_o}{p_1 - p_b} > \frac{P_2}{P_1} > 1 \tag{2 59}$$

from which it may be concluded that the pressure drop is more in the branch portion than in the main tube

Thus, it may be pointed out here that nature has provided frequent branchings in arteriole section not only to fecilitate blood supply to all parts of the body but also to help blood slow down in the capillary section Further, the rigidity of the walls, the non-Newtonian character of blood help to create greater resistance to flow causing a major pressure drop in the arteriolar section, McDonald (1974)

2 5 POWER LAW FLUID MODEL

In the previous sections, we have investigated the behaviour of blood by considering it as a Casson fluid In the following, we study the same characteristics of the blood flow by taking it as a powerlaw fluid under the same assumptions applicable to the same physical situation as described in Fig (2 1).

The stress-strain law for power law fluids is

$$\tau = -m \left|\frac{du}{dr}\right|^{n-1} \frac{du}{dr} \qquad \left(\frac{du}{dr} < 0\right) \tag{2 60}$$

where m and n are consistency and flow behaviour indices

Comparing the equations (2 8) and (2 60), we get

$$f(\tau) = (\tau/m)^{1/n}$$

Substituting the above expression of $f(\tau)$ in equation (2 10) and using equation (2 6) we get

$$Q = \left(-\frac{1}{2m}\frac{dp}{dz}\right)^{1/n} \frac{\pi R^{3+1/n}}{3+1/n} \tag{2 61}$$

Solving for (dp/dz) from (2 61), we get

$$\frac{dp}{dz} = \frac{-2m}{R^{3n+1}} \left[\frac{3n+1}{n\pi} Q\right]^{n} \tag{2 62}$$

Following a similar procedure as in the case of Casson model fluid, we get the differential equation for R as

$$\frac{dR}{dz} = \frac{-2m}{Eh\, R^{3n-1}} \left[\frac{3n+1}{n\pi} Q\right]^{n} \tag{2 63}$$

Integrating the above equation, with the initial condition

$$z = 0 \quad , \quad R = R_1 \tag{2 64}$$

We get R as

$$R = R_1 \left[1 - \frac{kz}{\ell}\right]^{1/3n} \tag{2 65}$$

where R_1 is given by the equation (2 24) and k is given by

$$k = \frac{6mn\ell}{Eh\, R_1^{3n}} \left[\frac{(3n+1)}{n\pi} Q\right]^{n} \tag{2 66}$$

The pressure drop across the artery is given by

$$p_1 - p_0 = \frac{Eh}{R_1} \left[\frac{1}{(1-k)^{1/3n}} - 1\right] \tag{2.67}$$

Hence the PR is obtained as

$$\lambda = \frac{Eh}{Q\,R_1} \left[\frac{1}{(1-k)^{1/3n}} - 1\right] \tag{2 68}$$

For $k << 1$, we get the PR for a powerlaw fluid flowing in an elastic tube, neglecting k^2 and higher powers in equation (2 68), as

$$\lambda = \frac{2m\,\ell}{Q\,R_1^{3n+1}} \left[\frac{3n+1}{n\pi}\,Q\right]^n \tag{2 69}$$

With the same approximation the PR for a rigid tube is given by

$$\lambda^* = \frac{2m\,\ell}{Q\,R_0^{3n+1}} \left[\frac{3n+1}{n\pi}\,Q\right]^n \tag{2 70}$$

Dividing equation (2 69) by equation (2 70) we obtain

$$\frac{\lambda}{\lambda^*} = \left(\frac{R_0}{R_1}\right)^{3n+1} \tag{2 71}$$

Since $R_0 < R_1$, it may be concluded from equation (2 71) that the PR of powerlaw fluid in an elastic pipe is less than that in a rigid pipe This implies that the PR increases as the artery becomes less flexible and this increase due to non-flexibility is enhanced further by pseudoplastic nature of the blood ($n < 1$)

2 6 EFFECTS OF BRANCHING

For the same geometry as in Fig (2 5) when the blood is considered as powerlaw fluid, we have from the equation (2 33) and

(2 65), the pressure drop in the main artery as follows

$$p_1 - p_b = \frac{Eh}{R_1}\left[\frac{1}{(1-k\beta)^{1/3n}} - 1\right] \qquad (2\ 72)$$

where k is given by (2 66)

This can be approximated for $k\beta << 1$ as

$$p_1 - p_b = \frac{2m\beta\ell}{R_1^{3n+1}}\left[\frac{3n+1}{n\pi}Q\right]^n \qquad (2\ 73)$$

The pressure drop in the jth branch is given by

$$p_b - p_o = \frac{2m(1-\beta)\ell}{R_j^{3n+1}}\left[\frac{3n+1}{n\pi}Q_j\right]^n \qquad (2\ 74)$$

Following the same procedure as in the case of Casson model, we get the pressure p_b at the junction as

$$p_b = \frac{\frac{p_1}{\beta} + \frac{p_o}{1-\beta}\left[\sum_1^M \alpha_j^{3+1/n}\right]^n}{\left(\frac{1}{\beta}\right) + \frac{1}{(1-\beta)}\left[\sum_1^M \alpha_j^{3+1/n}\right]^n} \qquad (2\ 75)$$

where

$$\alpha_j = R_j/R_1$$

Substituting the value of p_b from equation (2 75) in equation (2 73) the pressure drop for the system is obtained as

$$p_1 - p_o = \frac{2m\ell}{R_1^{3n+1}}\left[\frac{3n+1}{n\pi}Q\right]^n \left[\beta + \frac{1-\beta}{\{\Sigma\alpha_j^{3+1/n}\}^n}\right] \qquad (2\ 76)$$

The BF is given by

$$\lambda_B = \frac{2m\ell}{Q\,R_1^{3n+1}}\left[\frac{3n+1}{n\pi}\,Q\right]^n \left[\beta + \frac{1-\beta}{\left[\sum \alpha_j^{3+1/n}\right]^n}\right] \tag{2 77}$$

If all the branches are of equal radius, the equation (2 77) can be written after using equation (2 53) as

$$\lambda_B = \frac{2m\ell}{Q\,R_1^{3n+1}}\left[\frac{3n+1}{n\pi}\,Q\right]^n \left[\beta + \frac{M^{\frac{n+1}{2}}(1-\beta)}{d^{\frac{3n+1}{2}}}\right] \tag{2 78}$$

In case the main tube is also rigid, we have

$$\lambda_B^* = \frac{2m\ell}{Q\,R_o^{3n+1}}\left[\frac{3n+1}{n\pi}\,Q\right]^n \left[\beta + \frac{M^{\frac{n+1}{2}}(1-\beta)}{(d^*)^{\frac{3n+1}{2}}}\right] \tag{2 79}$$

where $d^* = M\alpha^{*2}$

We notice from the equation (2 78) that PR increases as the main vessel becomes less flexible It is also seen that the PR increases as the number of branches increases

2 7 CONCLUSIONS

The effects of branching and elasticity on the peripheral resistance for blood flow have been discussed by characterising the blood as Casson and powerlaw fluids It is shown that the elasticity decreases the resistance whereas the non-Newtonian character and branching enhance it

It is suggested that in the prearteriolar section, the rigidity of the tubes, the non-Newtonian character of the blood and

the frequent branchings, increase the resistance to flow, causing a major pressure drop Thus, the analysis presented here provides a theoretical explanation of the physiological observation, pointed out by McDonald (1974)

NOMENCLATURE

d	Ratio of the total area of cross section of all branches to the area of cross section of the main tube
E	Young's modulus of elasticity
h	Wall thickness of the blood vessel
ℓ	Length of the bed
M	Number of branches
m	Consistency index of powerlaw fluid
n	flow behaviour index of powerlaw fluid
p	pressure
p_b	pressure at the branching point
p_i	pressure at the entrance of the bed
p_o	pressure at the end point of the bed
Q	flux in the main tube
Q_j	flux in jth branch
R	Radius of the tube at axial distance z
R_i	Elongated Radius at the entrance of the bed ($z = 0$)
R_j	Radius of the jth branch
R_o	Unstretched Radius of the main tube
R_1	Starting value for iteration of R
R_2	first iterated value of R
u	axial velocity

α	Ratio of Radius of branch tube to main pipe when all the branches are of equal radius
α_j	R_j/R_i
β	proportional length of the main pipe to the total length of the bed
γ	R_o/R_i
η	Viscosity of the blood in the case of Casson model fluid
τ	Shear stress at any radial distance
τ_o	yield stress of Casson model fluid
τ_R	Wall shear
λ	Peripheral resistance, in the elastic pipe, of a non-Newtonian fluid
λ_N	Peripheral resistance of Newtonian fluid flowing in an elastic tube
λ^*	Peripheral resistance in the rigid tube of a Non-Newtonian fluid
λ_N^*	Peripheral resistance in the rigid tube of a Newtonian fluid
λ_B	Peripheral resistance due to a Non-Newtonian fluid flowing in a branched circuit where the main tube is elastic
λ_B^*	Peripheral resistance of a Non-Newtonian fluid flowing in a branched circuit having main pipe rigid
Δp	Pressure drop

REFERENCES

1 Batchelor, G K (1967)
An introduction to fluid dynamics
Cambridge University Press

2 Bergel, D H (Ed) (1972)
Cardiovascular fluid dynamics
Academic Press

3 Bird, R B , Stewart, W E and Lightfoot, E N (1960)
Transport phenomena
John Wiley and Sons, Inc

4 Burton, A C (1965)
Physiology and Biophysics of the circulation
Year Book Medical Publishers, Chicago

5 Caro, C G , Fitz-Gerald, J M , and Schroter, R C (1971)
Proc Roy Soc , Ser B, Vol 177, p 109

6 Casson, N (1958)
A flow equation for pigment oil suspensions of the printing ink type
In "Rheology of Disperse Systems" (Ed) C C Mills, Pergamon Press, p 84

7. Charm, S.E , and Kurland, G S (1962)
The flow behaviour and shear stress-shear rate characteristics of canine blood
Amer J Physiology, Vol 203, p 417

8. ____________ (1965)
Viscometry of human blood for shear rates of 0-100,000 Sec^{-1}
Nature, Vol 206, p 617

9 ____________ (1967)
Static method for determining blood yield stress
Nature, Vol 216, p 1121

10 ____________ (1972)
Blood Rheology
In "Cardiovascular Fluid Dynamics" (Ed)
Bergel, D H , Academic Press

11 Copley, A. L. and Scott-Blair, G W (1960)
Apparent viscosity and wall adherence of blood vessels
In "Flow Properties of blood and other Biological Systems" (ed) A Copley and G Stainby, Pergamon Press, p 97

12 Cokelet et al (1963)
Rheology of human blood-measurement near and at zero shear rate
Trans Soc Rheol , Vol 7, p 303

13 Cox, R H (1968)
Wave propagation through a Newtonian fluid contained in a thick-walled visco-elastic tube
Biophys J , Vol 8, p 691

14 Fung, Y C (1966)
Biomechanics
ASME

15 Hershey, D and Cho, S J (1966)
Laminar flow of suspensions (blood) Thickness and effective velocity of the film adjacent to the wall
In "Chemical Engineering in Medicine"
Chemical Engineering Progress Symposium Series No 62, (ed) E F Leonard, A I Ch E., Philadelphia, p 139

16 Huckaba, et al (1968)
A generalised approach to the modelling of arterial Blood flow
Bull Math. Biophys , Vol 30, p 645

17 Krovetz, L.J (1965)
The effect of vessel branching on haemodynamic stability
Phys Med Biol , Vol 10, p 417

18. Lightfoot, E.N (1974)
Transport phenomena and living systems
Wiley Interscience, New York

19. McDonald, D.A , (1974)
Blood Flow in arteries
Baltimore Williams and Wilkins Co

20 Merrill, et al (1963)
Non-Newtonian Rheology of human blood Effect of fibrinogen deduced by subtraction
Circulation Res , Vol 13, p 48.

21 Merrill et al (1963)
Rheology of human blood and red cell plasma membrane
Bibli Anat ,Vol 4, p 51

22 ______________ (1965)
The Casson equation and rheology of blood near shear zero
In "Proceedings Fourth international Congress on Rheology", part 4 (ed) Copley A.L ,
Interscience Publishers, p 135

23 Middleman, S (1972)
Transport phenomena in the Cardiovascular system
Wiley-Interscience, New York

24. Mirsky, I (1967)
Wave Propagation in a viscous fluid contained in an orthotropic elastic tube
Biophys J Vol 7, p 165.

25. Morgan, G W (1952)
On the study of laminar flow of a viscous incompressible fluid in an elastic tube
Bull Math. Biophys , Vol. 14, p 19

26 Rashevsky, N (1945)
A problem in the Mathematical Biophysics of blood-circulation. II-Relation between pressure and flow of a viscous fluids in an elastic distensible tube
Bull. Math. Biophysics 7, 35 - 39

27 Reiner and Scott-Blair (1959)
The flow of blood through narrow tubes
Nature, Vol. 184, p 354

28. Skalak, R. (1972)
Synthesis of Complete Circulation
In "Cardiovascular fluid dynamics (ed) D.H Bergel
Academic Press, p 341

29. Whitmore, R L (1968)
Rheology of Circulation
Pergamon Press

30 Womersley, J R (1955)
Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known.
J. Physiol. Vol. 127, p 553

31 Womersley, J R (1957)
An elastic tube theory of pulse transmission and oscillatory flow in mammalian arteries
Wright air Development Centre,
WADC Report TR 56-614

32 ______________ (1958)
Oscillatory flow in Arteries The constrained elastic tube as a model of arterial flow in pulse transmission
Phys Med Biol Vol. 2,p 178.

CHAPTER III

SWIMMING OF SPERMATOZOA IN A CERVICAL CANAL

3 1 INTRODUCTION

Swimming of spermatozoa in human genital organ has received much attention in recent times, Reynolds (1965), Shack and Lardner (1974), Smelser et al (1974), Lighthill (1974, 1975), Pironneau (1975) The spermatozoa are living cells (length 5×10^{-3} cm, diameter 10^{-5} cm) having their own motility and move towards the oviduct through the mucus filling the cervical canal by sending waves of lateral displacement down the tail or flagellum A mathematical analysis of swimming of spermatozoa was first carried out by Taylor (1951) Later, Hancock (1953) studied the self propulsion of microscopic organisms through liquids Reynolds (1965) discussed a two dimensional model by assuming that the amplitude of the tail motion is small as compared to the distance between the walls Shack and Lardner (1974) has also studied this problem under a long wave length approximation However, none of these studies has taken into account the special structure of the mucus and the dynamical interaction of the cervical canal

Cervical mucus is a suspension of macromolecules, having higher molecular weight, in a water like liquid. The viscosity of the suspension is 0 03 P, Odeblad (1959, 1962) In the luteal phase the mucus resembles like a close mesh, with macromolecules spreading

all over the canal, having a spacing of 0 03 μm in them, Odeblad (1968), Elstein (1971), Davajan et al (1971) In the mid cycle period, the molecules group together to form micelles which align themselves along the walls of the cervical canal, Odeblad (1968)

The role played by cervix in spermatozoa transport has been reviewed by Moghissi (1969, 1971) It has been pointed out that, "the transport of sperm through the cervical mucus is too rapid to be accounted for by the in vitro swimming speed of approximately 50 μ sec of sperm in cervical mucus", Smelser et. al (1974) Various suggestions have been put forward to explain this rapid passage of spermatozoa by keeping in view the structure of the mucus, the alignment of the micelles along the wall, the possible interactions of the sperm and the wave generated by it with micelles, Odeblad (1962, 1971), Davajan et al (1970) A hydrodynamical analysis has also been presented to explain the rapid motion of the sperm by assuming that, due to the above mentioned interaction, the wall of the cervix undergoes a peristaltic wave motion, Smelser et al (1974)

Here, in this chapter, a different approach to explain this rapid movement of spermatozoa is suggested by taking into account the above mentioned dynamical interaction It may be noted that the wave generated by the sperm might interact with the micelles and produce a force in the direction of the motion of spermatozoa This force may be caused by the flexibility, unevenness and the vibration of the micelles

Further, due to the concentration of micelles near the walls in the midcycle period, the viscosity of the mucus is greater near the walls in comparison to that at the centre This creates comparitively more viscous resistance near the wall and causes the sperm to swim only in the central region of the cervix This resistance also slows down the motion of fluid near the cervical wall which in turn causes the sperm to move faster

Keeping the above in view, we study here the motion of spermatozoa by taking into account the following aspects

(i) The effect of force, caused due to the dynamical interaction of the micelles with the swimming sperm, by assuming that it is proportional to the shear stress at the wall

(ii) The effect of a more viscous peripheral layer due to concentration and alignment of the micelles near the wall in the midcycle phase

3.2 MATHEMATICAL ANALYSIS

Consider the motion of a spermatozoon in the cervical canal whose physical configuration is shown in figure (3 1) The spermatozoon is assumed to be an infinite sheet which propels itself forward by passing sinusoidal waves down itself in the backward direction

Let V_p be the propelling velocity of the sheet in the negative x-direction In a frame moving with the sheet, the form of the sheet is given by

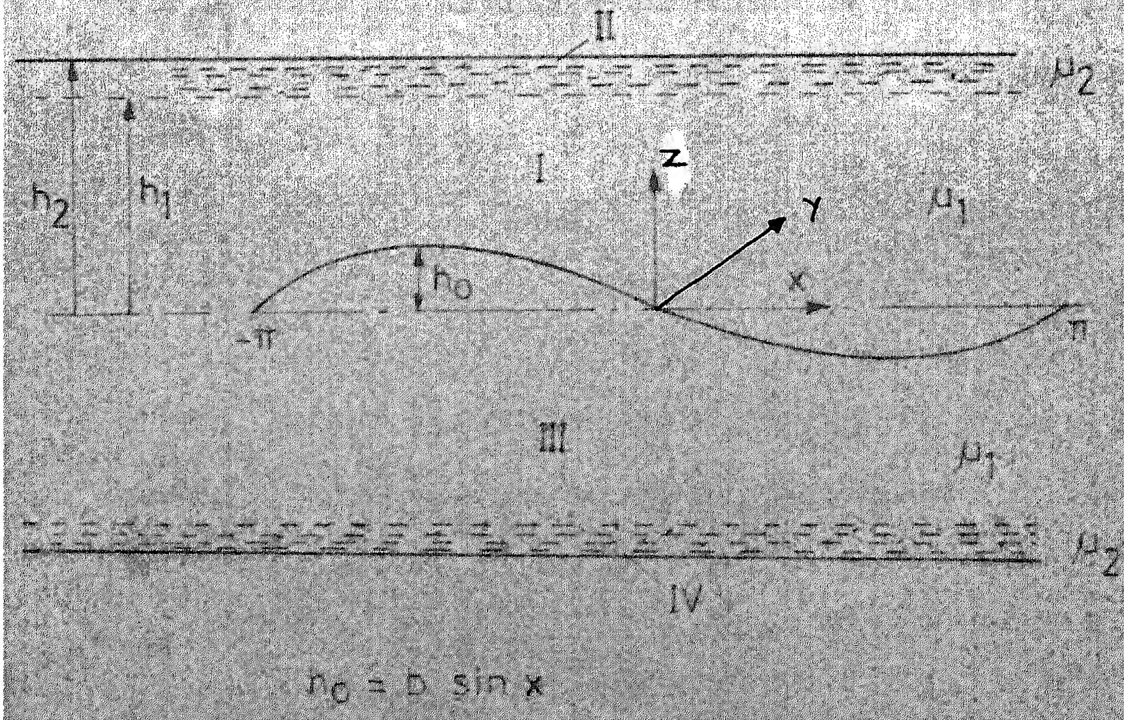

FIG. 3.1 MOTION OF SPERMATOZOON IN A LAYERED MEDIUM

$$z = b \sin[k(x - ct)] = h_o \qquad (3.1)$$

where b is the amplitude and c is the velocity of the wave. In a frame moving with a velocity $(c-V_p)$ in the positive direction, the sheet appears to be stationary and the fluid flow steady. Since the sheet is performing sinusoidal waves, the velocity and pressure distribution are not symmetric about $y = 0$. In addition, as pointed out earlier and shown in fig. (3.1) a more viscous peripheral layer exists near the walls in the midcycle phase. Hence, the entire region between the walls is divided into four parts two above the sheet and two below it.

Let p^+, p^- be the pressure distribution above and below the sheet and (u_1^+, v_1^+, w_1^+), (u_2^+, v_2^+, w_2^+), (u_1^-, v_1^-, w_1^-) and (u_2^-, v_2^-, w_2^-) are the velocity components of the fluid respectively in the regions I, II, III and IV. Neglecting the inertia effects, as Reynolds number is very small (10^{-3}), the equations of motion in the respective regions are given as follows.

(I) In the region, $h_o < z < h_1$

$$- \frac{\partial p^+}{\partial x} + \mu_1 \left[\frac{\partial^2 u_1^+}{\partial x^2} + \frac{\partial^2 u_1^+}{\partial y^2} + \frac{\partial^2 u_1^+}{\partial z^2} \right] = 0$$

$$- \frac{\partial p^+}{\partial y} + \mu_1 \left[\frac{\partial^2 v_1^+}{\partial x^2} + \frac{\partial^2 v_1^+}{\partial y^2} + \frac{\partial^2 v_1^+}{\partial z^2} \right] = 0 \qquad (3.2\text{-a})$$

$$\frac{\partial u_1^+}{\partial x} + \frac{\partial v_1^+}{\partial y} + \frac{\partial w_1^+}{\partial z} = 0$$

(II) In the region, $h_1 < z < h_2$

$$-\frac{\partial p^+}{\partial x} + \mu_2 \left[\frac{\partial^2 u_2^+}{\partial x^2} + \frac{\partial^2 u_2^+}{\partial y^2} + \frac{\partial^2 u_2^+}{\partial z^2}\right] = 0$$

$$-\frac{\partial p^+}{\partial y} + \mu_2 \left[\frac{\partial^2 v_2^+}{\partial x^2} + \frac{\partial^2 v_2^+}{\partial y^2} + \frac{\partial^2 w_2^+}{\partial z^2}\right] = 0 \qquad (3\ 2\text{-b})$$

$$\frac{\partial u_2^+}{\partial x} + \frac{\partial v_2^+}{\partial y} + \frac{\partial w_2^+}{\partial z} = 0$$

(III) In the region, $-h_1 < z < h_o$, the equations of motion and continuity are the same as in (I) where p^+, (u_1^+, v_1^+, w_1^+) are replaced respectively by p^-, (u_1^-, v_1^-, w_1^-)

(IV) In the region, $-h_2 < z < -h_1$, the equations of motion and continuity in (II) hold good where p^+, (u_2^+, v_2^+, w_2^+) are replaced by p^-, (u_2^-, v_2^-, w_2^-) respectively

The boundary conditions are

(i)
$$u_1^{\pm} = -c$$

$$v_1^{\pm} = 0 \qquad z = h_o \qquad (3\ 3\text{-a})$$

$$w_1^{\pm} = -bkc \cos ky$$

(ii)
$$u_2^{\pm} = V_p - c \pm \frac{\mu_2}{\beta'} \left(-\frac{\partial u_2^{\pm}}{\partial z}\right)$$

$$v_2^{\pm} = \pm \frac{\mu_2}{\beta'} \left(-\frac{\partial v_2^{\pm}}{\partial z}\right) \qquad z = \pm h_2 \qquad (3\ 3\text{-b})$$

$$w_2^{\pm} = \pm w$$

(iii) $u_1^{\pm} = u_2^{\pm}$

$$v_1^{\pm} = v_2^{\pm}$$

$$\mu_1 \frac{\partial u_1^{\pm}}{\partial z} = \mu_2 \frac{\partial u_2^{\pm}}{\partial z} \qquad z = \pm h_1 \qquad (3\ \text{.-c})$$

$$\mu_1 \frac{\partial v_1^{\pm}}{\partial z} = \mu_2 \frac{\partial v_2^{\pm}}{\partial z}$$

where β' is t e constant of proportionality and is negative because the force exerted by vibrating micelles is assumed to be proportional to the shear stress at the wall and acts in the direction of motion of the sperm (boundary conditions (ii)) The boundary conditions (iii) are simply the continuity of the velocity and shear stress in x and y directions at the interface of two fluids whose viscosities are μ_1 and μ_2 ($\geq \mu_1$)

Non-dimensionalising the variables and parameters as below,

$$x_* = kx,\ y = ky,\ z_* = z/h_2$$

$$p_*^{\pm} = p^{\pm}\ k\ h_2^2/\mu_1\ c$$

$$u_{i_*}^{\pm} = u_i^{\pm}/c,\ v_{i_*}^{\pm} = v_i^{\pm}/c,\ w_{i_*}^{\pm} = w_i^{\pm}/kch_2 \quad (i = 1,2)$$

$$V_{p_*} = V_p/c,\ Q_{x_*} = Q_x/ch_2,\ Q_{y_*} = Q_y/ch_2\ ,\ \bar{\mu} = \frac{\mu_1}{\mu_2}\ ,$$

$$h_{1_*} = h_1/h_2,\ \beta = kh_2\ ,\ \beta'_* = \beta'\ h_2/\mu_1\ ,\ b_* = b/h_2$$

we get the above equations of motion as follows (after dropping the '*' notation for convenience) :

(I) In the region, $h_o < z < h_1$, $(h_o = b \sin x)$

$$-\frac{\partial p^+}{\partial x} + [\beta^2 \frac{\partial^2 u_1^+}{\partial x^2} + \beta^2 \frac{\partial^2 u_1^+}{\partial y^2} + \frac{\partial^2 u_1^+}{\partial z^2}] = 0$$

$$-\frac{\partial p^+}{\partial y} + [\beta^2 \frac{\partial^2 v_1^+}{\partial x^2} + \beta^2 \frac{\partial^2 v_1^+}{\partial y^2} + \frac{\partial^2 v_1^+}{\partial z^2}] = 0$$

$$\frac{\partial u_1^+}{\partial x} + \frac{\partial v_1^+}{\partial y} + \frac{\partial w_1^+}{\partial z} = 0 \qquad (3\ 4\ a)$$

(II) In the region, $h_1 < z < 1$

$$-\bar{\mu} \frac{\partial p^+}{\partial x} + [\beta^2 \frac{\partial^2 u_2^+}{\partial x^2} + \beta^2 \frac{\partial^2 u_2^+}{\partial y^2} + \frac{\partial^2 u_2^+}{\partial z^2}] = 0$$

$$-\bar{\mu} \frac{\partial p^+}{\partial y} + [\beta^2 \frac{\partial^2 v_2^+}{\partial x^2} + \beta^2 \frac{\partial^2 v_2^+}{\partial y^2} + \frac{\partial^2 v_2^+}{\partial z^2}] = 0$$

$$\frac{\partial u_2^+}{\partial x} + \frac{\partial v_2^+}{\partial y} + \frac{\partial w_2^+}{\partial z} = 0 \qquad (3\ 4\text{-}b)$$

Similar equations with superscript '−' may be written for the regions III and IV Using long wave length approximation, i e $\beta << 1$, and assuming an asymptotic expansion for the solution, as in Shack and Lardner (1974), we get the equations of motion involving lowest order terms as follows

In the region, $h_o < z < h_1$

(I)

$$-\frac{\partial p^+}{\partial x} + \frac{\partial^2 u_1^+}{\partial z^2} = 0$$

$$-\frac{\partial p^+}{\partial y} + \frac{\partial^2 v_1^+}{\partial z^2} = 0 \qquad (3\ 5\text{-a})$$

$$\frac{\partial u_1^+}{\partial x} + \frac{\partial v_1^+}{\partial y} + \frac{\partial w_1^+}{\partial z} = 0$$

(II) In the region, $h_1 < z < 1$

$$-\bar{\mu}\ \frac{\partial p^+}{\partial x} + \frac{\partial^2 u_2^+}{\partial z^2} = 0$$

$$-\bar{\mu}\ \frac{\partial p^+}{\partial y} + \frac{\partial^2 v_2^+}{\partial z^2} = 0 \qquad (3\ 5\text{-b})$$

$$\frac{\partial u_2^+}{\partial x} + \frac{\partial v_2^+}{\partial y} + \frac{\partial w_2^+}{\partial z} = 0$$

(III) In the region, $-h_1 < z < h_o$

$$-\frac{\partial p^-}{\partial y} + \frac{\partial^2 u_1^-}{\partial z^2} = 0$$

$$-\frac{\partial p^-}{\partial y} + \frac{\partial^2 v_1^-}{\partial z^2} = 0 \qquad (3\ 5\text{-c})$$

$$\frac{\partial u_1^-}{\partial x} + \frac{\partial v_1^-}{\partial y} + \frac{\partial w_1^-}{\partial z} = 0$$

(IV) In the region, $-1 < z < -h_1$

$$-\bar{\mu}\ \frac{\partial p^-}{\partial x} + \frac{\partial^2 u_2^-}{\partial z^2} = 0$$

$$-\bar{\mu}\ \frac{\partial p^-}{\partial y} + \frac{\partial^2 v_2^-}{\partial z^2} = 0 \qquad (3\ 5\text{-d})$$

$$\frac{\partial u_2^-}{\partial x} + \frac{\partial v_2^-}{\partial y} + \frac{\partial w_2^-}{\partial z} = 0$$

where $\bar{\mu} = \frac{\mu_1}{\mu_2} < 1$

The non-dimensionalised boundary conditions are

(i) $u_1^{+} = -1$

$v_1^{\pm} = 0$ on $z = b \sin x = 1_o$ (3 6-a)

$w_1^{\pm} = -b \cos x$

(ii) $u_2^{\pm} = V_p - 1 \pm \alpha\left(-\frac{\partial u_2^{\pm}}{\partial z}\right)$

$v_2^{\pm} = \pm \alpha \left(-\frac{\partial v_2^{\pm}}{\partial z}\right)$ $z = \pm 1$ (3 6-b)

$w_2^{\pm} = \pm w$

(iii) $u_1^{\pm} = u_2^{\pm}$

$v_1^{\pm} = v_2^{\pm}$ $z = \pm h_1$

$\bar{\mu} \frac{\partial u_1^{\pm}}{\partial z} = \frac{\partial u_2^{\pm}}{\partial z}$ (3 6-c)

$\bar{\mu} \frac{\partial v_1^{\pm}}{\partial z} = \frac{\partial v_2^{\pm}}{\partial z}$

where $\alpha = \frac{1}{\beta'\bar{\mu}}$ and is negative

Solving the equations (3 5-a), (3 5-b), (3 5-c) and (3 5-d) in the regions above and below the sheet and using the boundary conditions (3 6), we get

$$u_1^{\pm} = \frac{1}{2} \frac{\partial p^{\pm}}{\partial x} z^2 + A_1^{\pm} z + B_1^{\pm} \tag{3 7}$$

$$u_2^{\pm} = \frac{\bar{\mu}}{2} \frac{\partial p^{\pm}}{\partial x} z^2 + A_2^{\pm} z + B_2^{\pm} \tag{3 8}$$

$$v_1^{\pm} = \frac{1}{2}\,\frac{\partial p^{\pm}}{\partial y}\,z^2 + C_1^{\pm}\,z + D_1^{\pm} \tag{3 9-a}$$

$$v_2^{\pm} = \frac{\bar{\mu}}{2}\,\frac{\partial p^{\pm}}{\partial y}\,z^2 + C_2^{\pm}\,z + D_2^{\pm} \tag{3 9-b}$$

where

$$A_1^{\pm} = \frac{\pm V_P \mp \frac{1}{2}\frac{\partial p^{\pm}}{\partial x}\left[(1-h_o^2+2\alpha) + (\bar{\mu}-1)(1-h_1^2+2\alpha)\right]}{(1 \mp h_o + \alpha) + (\bar{\mu}-1)(1-h_1+\alpha)} \tag{3 10-a}$$

$$A_2^{\pm} = \bar{\mu}\,A_1^{\pm} \tag{3 10-b}$$

$$C_1^{\pm} = \frac{\mp\frac{1}{2}\frac{\partial p^{\pm}}{\partial y}\left[(1-h_o^2+2\alpha) + (\bar{\mu}-1)(1-h_1^2+2\alpha)\right]}{(1 \mp h_o + \alpha) + (\bar{\mu}-1)(1-h_1+\alpha)} \tag{3 10-c}$$

$$C_2^{\pm} = \bar{\mu}\,C_1^{\pm} \tag{3 10-d}$$

$$B_1^{\pm} = -1 - \frac{1}{2}\,\frac{\partial p^{\pm}}{\partial x}\,h_o^2 - A_1^{\pm}\,h_o \tag{3 11-a}$$

$$B_2^{\pm} = V_P - 1 - \bar{\mu}\,\frac{\partial p^{\pm}}{\partial x}\left(\frac{1}{2}+\alpha\right) \mp A_2^{\pm}\,(1+\alpha) \tag{3 11-b}$$

$$D_1^{\pm} = -\frac{1}{2}\,\frac{\partial p^{\pm}}{\partial y}\,h_o^2 - C_1^{\pm}\,h_o \tag{3 11-c}$$

$$D_2^{\pm} = -\bar{\mu}\,\frac{\partial p^{\pm}}{\partial y}\left(\frac{1}{2}+\alpha\right) \mp C_2^{\pm}\,(1+\alpha) \tag{3 11-d}$$

It may be noted that the equations (3 10-a) to (3 11-d) contain the unknown velocity V_P, which is to be determined by using the equation of force balance on the sheet

The flux $Q_x^{\pm}$, $Q_y^{\pm}$ in the x and y directions above and below the sheet are given by

$$Q_x^+ = \int_{b \sin x}^{h_1} u_1^+ \, dz + \int_{h_1}^{1} u_2^+ \, dz \qquad (3.12\text{-a})$$

$$Q_x^- = \int_{-1}^{-h_1} u_2^- \, dz + \int_{-h_1}^{b \sin x} u_1^- \, dz \qquad (3.12\text{-b})$$

$$Q_y^+ = \int_{b \sin x}^{h_1} v_1^+ \, dz + \int_{h_1}^{1} v_2^+ \, dz \qquad (3.13\text{-a})$$

$$Q_y^- = \int_{-1}^{-h_1} v_2^- \, dz + \int_{-h_1}^{b \sin x} v_1^- \, dz \qquad (3.13\text{-b})$$

Substituting the expressions for velocity components from equations (3.7) to (3.9-b) and integrating, we obtain the fluxes as

$$Q_x^{\pm} = \frac{G^{\pm}}{F^{\pm}} \left[-\frac{1}{12} \frac{\partial p^{\pm}}{\partial x} \right] + \frac{H^{\pm}}{F^{\pm}} \qquad (3.14)$$

$$Q_y^{\pm} = \frac{G^{\pm}}{F^{\pm}} \left[-\frac{1}{12} \frac{\partial p^{\pm}}{\partial y} \right] \qquad (3.15)$$

where

$$F^{\pm} = (1 \mp h_o + \alpha) + (\bar{\mu} - 1)(1 - h_1 + \alpha) \qquad (3.16\text{-a})$$

$$G^{\pm} = [(1 \mp h_o)^2 + (\bar{\mu} - 1)(1 - h_1)^2]^2$$
$$+ 4(\bar{\mu} - 1)(1 \mp h_o)(1 - h_1)(h_1 \mp h_o)^2$$
$$+ 4\alpha\bar{\mu} \, [(1 \mp h_o)^3 + (\bar{\mu} - 1)(1 - h_1)^3] \qquad (3.16\text{-b})$$

$$H^{\pm} = \frac{V_p}{2} \; [(1 \mp h_o)^2 + (\bar{\mu} - 1)\,(1 - h_1)^2\,]$$

$$- (1 \mp h_o)\; [(1 \mp h_o + \alpha) + (\bar{\mu} - 1)(1 - h_1 + \alpha)\,] \quad (3\ 16\text{-c})$$

Integrating the continuity equation between $z = h_o$ and $z = 1$, we get

$$\frac{\partial}{\partial x}(Q_x^+) + \frac{\partial}{\partial y}(Q_y^+) = 0 \qquad (3\ 17)$$

Substituting for Q_x^+ and Q_y^+, we get the equation governing the pressure in the region above the sheet as

$$\frac{\partial}{\partial x}\left[-\frac{1}{12}\;\frac{G^+}{F^+}\frac{\partial p^+}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{-1}{12}\;\frac{G^+}{F^+}\frac{\partial p^+}{\partial y}\right]$$

$$+ \frac{\partial}{\partial x}\left[\frac{H^+}{F^+}\right] = 0 \qquad (3\ 18)$$

where F^+, G^+, H^+ are given by equation (3 16-a) to (3 16-c)

Similarly the equation in the region below the sheet is obtained as follows

$$\frac{\partial}{\partial x}\left[\frac{-1}{12}\frac{G^-}{F^-}\frac{\partial p^-}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{-1}{12}\frac{G^-}{F^-}\frac{\partial p^-}{\partial y}\right] + \frac{\partial}{\partial x}\left[\frac{H^-}{F^-}\right] = 0 \qquad (3\ 19)$$

where F^-, G^-, H^- are given by equation (3 16-a) to (3 16-c)

It is pointed out here that these equations may be seen as the generalized forms of Reynolds lubrication equation applicable to spermatozoa motion, Cameron (1966)

Particular Cases

(1) When $\mu_1 = u_2$ and $\alpha \neq 0$, we get the following Reynolds equation in two dimensions for the case of a fluid of constant viscosity in which the sperm is swimming and having the dynamical interaction proportional to the shear stress at the wall,

$$\frac{\partial}{\partial x}\left[-\frac{1}{12}\,\frac{G_1^{\pm}}{F_1^{\pm}}\,\frac{\partial p^{\pm}}{\partial x}\right]+\frac{\partial}{\partial y}\left[-\frac{1}{12}\,\frac{G_1^{\pm}}{F_1^{\pm}}\,\frac{\partial p^{\pm}}{\partial y}\right]+\frac{\partial}{\partial x}\left[\frac{H_1^{\pm}}{F_1^{\pm}}\right]=0 \qquad (3\ 20)$$

where

$$F_1^{\pm} = (1 \mp h_o + \alpha) \qquad (3\ 21\text{-a})$$

$$G_1^{\pm} = (1 \mp h_o)^3\ (4\alpha + 1 \mp h_o) \qquad (3\ 21\text{-b})$$

$$H_1^{\pm} = \frac{V_P}{2}(1 \mp h_o)^2 - (1 \mp h_o)\ (1 \mp h_o + \alpha) \qquad (3\ 21\text{-c})$$

In one dimensional case, the above equation reduces to

$$\left[\frac{-G_1^{\pm}}{12F_1^{\pm}}\right]\frac{\partial p^{\pm}}{\partial x} + \frac{H_1^{\pm}}{F_1^{\pm}} = Q \qquad (3\ 22)$$

where Q, a constant of integration, as can be seen from equation of continuity in this case represents the flux

(2) When $\alpha = 0$ i e , there is no dynamical interaction, the equation governing the pressure in the canal when the viscosity of the mucus varies as a step function, is given by

$$\frac{\partial}{\partial x}\left[\frac{-1}{12}\frac{G_2^{\pm}}{F_2^{\pm}}\frac{\partial p^{\pm}}{\partial x}\right]+\frac{\partial}{\partial y}\left[\frac{-1}{12}\frac{G_2^{\pm}}{F_2^{\pm}}\frac{\partial p^{\pm}}{\partial y}\right]+\frac{\partial}{\partial x}\left[\frac{H_2^{\pm}}{F_2^{\pm}}\right]=0 \qquad (3\ 23)$$

where

$$F_2^{\pm}=(1\mp h_o)+(\bar{\mu}-1)(1-h_1) \qquad (3\ 24\text{-a})$$

$$G_2^{\pm}=[(1\mp h_o)^2+(\bar{\mu}-1)(1-h_1)^2]^2$$

$$+4(\bar{\mu}-1)(1\mp h_o)(1-h_1)(h_1\mp h_o)^2 \qquad (3\ 24\text{-b})$$

$$H_2^{\pm}=\frac{V_P}{2}[(1\mp h_o)^2+(\bar{\mu}-1)(1-h_1)^2]$$

$$-(1\mp h_o)[(1\mp h_o)+(\bar{\mu}-1)(1-h_1)] \qquad (3\ 24\text{-c})$$

In one dimensional case this equation becomes

$$\left[\frac{-1}{12}\frac{G_2^{\pm}}{F_2^{\pm}}\right]\frac{\partial p^{\pm}}{\partial x}+\frac{H_2^{\pm}}{F_2^{\pm}}=Q \qquad (3\ 25)$$

where Q is again a constant of integration and represents the flux

It is interesting to note that when $\alpha=0$ and $\mu_1=\mu_2$ both the particular cases obtained above reduce to

$$-\frac{(1\mp h_o)^3}{12}\frac{\partial p^{\pm}}{\partial x}+\left(\frac{V_P}{2}-1\right)(1\mp h_o)=Q \qquad (3\ 26)$$

which coincides with the equation obtained by Shack and Lardner (1974)

In what follows, we consider one dimensional case of swimming of spermatozoa

Since the sheet is self propelling the resultant fluid force acting on it must be zero. This is expressed as

$$\int (T^{+} - T^{-})\, ds = 0 \tag{3.27}$$

where T^{+}, T^{-} are the fluid forces on the upper and lower portions of the sheet. Substituting the expressions for shear, from stress-strain law and taking the long wave length approximation ($\beta << 1$), the equation (3.27) reduces to

$$\int_{-\pi}^{\pi} [p]\, dx = 0 \tag{3.28}$$

$$\int_{-\pi}^{\pi} \left\{ \left[\frac{\partial u}{\partial z}\right]_{z=b} \sin x + b \cos x\, [p] \right\} dx = 0 \tag{3.29}$$

where [ ] denote the difference of the enclosed quantity evaluated above and below the sheet. Appropriate expression for $\left[\frac{\partial u}{\partial z}\right]$ has to be substituted for getting the equation of balance of fluid force on the sheet.

The rate of work done by the swimming sheet against viscous forces per unit thickness is

$$W = \int n_j \,(\tau^{+}_{ji} - \tau^{-}_{ji})\, u_i \, ds \tag{3.30}$$

where n_j is the unit normal to the sheet, u_i the velocities of particles on the sheet in the fixed frame, and $\tau^{\pm}_{ji}$ are the fluid stresses on the upper and lower sides of the sheet.

Substituting for the shear stresses and taking the long wave length approximation we get,

$$W = \int_{-\pi}^{\pi} - b\ [p]\ \cos x\ dx \qquad (3\ 31)$$

3 3 EFFECT OF DYNAMICAL INTERACTION

In this section we investigate the effect of dynamical interaction of the micelles at the walls on the motion of sperm in cervical canal The differential equation for determining the pressure is given by from equation (3 22),

$$\frac{dp^{\pm}}{dx} = \frac{-12Q\ (\alpha + 1 \mp h_o)}{(4\alpha + 1 \mp h_o)\ (1 \mp h_o)^3} + \frac{6V_p\ (1 \mp h_o) - 12(\alpha + 1 \mp h_o)}{(4\alpha + 1 \mp h_o)(1 \mp h_o)^2} \qquad (3\ 32)$$

where the parameter α determines the dynamical interaction and is a negative constant as pointed out earlier.

Integrating equation (3 32) and expanding the resultant in terms of $\alpha(|\alpha| << 1)$, we have after neglecting α^2 and its higher powers,

$$\Delta p_\lambda = -\ 12Q\,\pi\left[\frac{(b^2 + 2)}{(1-b^2)^{5/2}} - \frac{3\alpha(3b^2 + 2)}{(1-b^2)^{7/2}}\right] + 12\ V_p\ \pi\ \left[\frac{1}{(1-b^2)^{3/2}} - \frac{2\alpha\ (b^2 + 2)}{(1 - b^2)^{5/2}}\right] - 12\,\pi\left[\frac{2}{(1-b^2)^{3/2}} - \frac{3\alpha\ (b^2 + 2)}{(1-b^2)^{5/2}}\right] \qquad (3\ 33)$$

Where $\Delta p_\lambda = p(\pi) - p(-\pi)$

The force balance equation (3 29) in this case is given by

$$\int_{-\pi}^{\pi}\{[\frac{\partial u}{\partial z}]_{z\,=\,b\,\sin x} + b\ \cos x\ [p]\}dx = 0 \qquad (3\ 29\text{-a})$$

Substituting for $[\frac{\partial u}{\partial z}]$ from the equations (3 7) in equation (3 29-a), we get, after simplification,

$$\frac{8\pi\, V_P}{\sqrt{(1+\alpha)^2 - b^2}} = 2\Delta p_\lambda + \int_{-\pi}^{\pi} b\ \sin x\ [\frac{dp}{dx}]\ dx$$

$$+ \alpha \int_{-\pi}^{\pi} [\frac{(1-h_o)\ dp^+/dx}{\alpha + 1 - h_o} + \frac{(1+h_o)\ dp^-/dx}{\alpha + 1 + h_o}]\ dx \qquad (3\ 34)$$

Substituting for $\frac{dp^+}{dx}$, $\frac{dp^-}{dx}$ and evaluating the integrals, we get from equation (3 34), after expanding in powers of α, and neglecting α^2 and higher powers,

$$8\pi\, V_P\, [\frac{1}{(1-b^2)^{1/2}} - \frac{\alpha}{(1-b^2)^{3/2}}]$$

$$= 2\Delta p_\lambda - 24Q\ \pi[\frac{3b^2}{(1-b^2)^{5/2}} - \frac{\alpha(4b^4 + 13b^2 - 2)}{(1-b^2)^{7/2}}]$$

$$+ 24\ V_P\ \pi[\frac{b^2}{(1-b^2)^{3/2}} - \frac{\alpha(7b^2 - 1)}{(1-b^2)^{5/2}}]$$

$$- 24\pi\, [\frac{2b^2}{(1-b^2)^{3/2}} - \frac{\alpha(11b^2 - 2)}{(1-b^2)^{3/2}}] \qquad (3\ 35)$$

Solving the equations (3 33) and (3 35) for Q and V_P, we get

$$Q = \frac{(1-b^2)^{3/2}\quad \Delta p_\lambda}{12\pi} + \frac{(b^2-1)}{1+2b^2} - \frac{6b^2\ \alpha}{(1-b^2)(1+2b^2)^2} \qquad (3\ 36)$$

$$V_F = \frac{\Delta p_\lambda \,(1-b^2)^{1/2}}{4\pi} + \frac{3b^2}{(1+2b^2)} + \frac{\Delta p_\lambda \quad \alpha \quad (2-b^2)}{4\pi \,(1-b^2)^{1/2}} - \frac{6\alpha\, b^2}{(1+2b^2)^2} \qquad (3\ 37)$$

When $\alpha = 0$, there is no reaction at the wall The result coincides with that of Shack and Lardner (1974) Since $\alpha \leq 0$, in the free channel case, when $\Delta p_\lambda = 0$, we notice that the effect of dynamical interaction is to enhance the propelling velocity of the sperm by an amount $\frac{6\alpha\, b^2}{(1+2b^2)^2}$

Substituting the expression for $[\frac{dp}{dx}]$ and using the values of Q and V_P, we get the work done, from equation (3 31) as follows

$$W = \frac{24\pi\, b^2}{(1-b^2)^{1/2}\,(1+2b^2)} - \frac{72\,\alpha\, b^2}{(1-b^2)^{3/2}\,(1+2b^2)^2} \qquad (3\ 38)$$

For small amplitudes with $\Delta p_\lambda = 0$, the expressions for V_F, Q and W can be approximated as,

$$\left.\begin{aligned} V_F &= 3b^2 - 6\,\alpha\, b^2 \\ Q &= -1 + 3b^2 - 6\alpha\, b^2 \\ W &= 24\pi\, b^2 - 72\alpha\, b^2 \end{aligned}\right] \qquad (3\ 39)$$

Which agree with Shack and Lardner (1974), when $\alpha = 0$

It is seen here that since $\alpha \leq 0$, both V_P and W increase as the force due to dynamical interaction increases This increase is further enhanced by the increase in the amplitude of the wave motion (i e b) These results can also be seen from the figures (3 2) and (3 3)

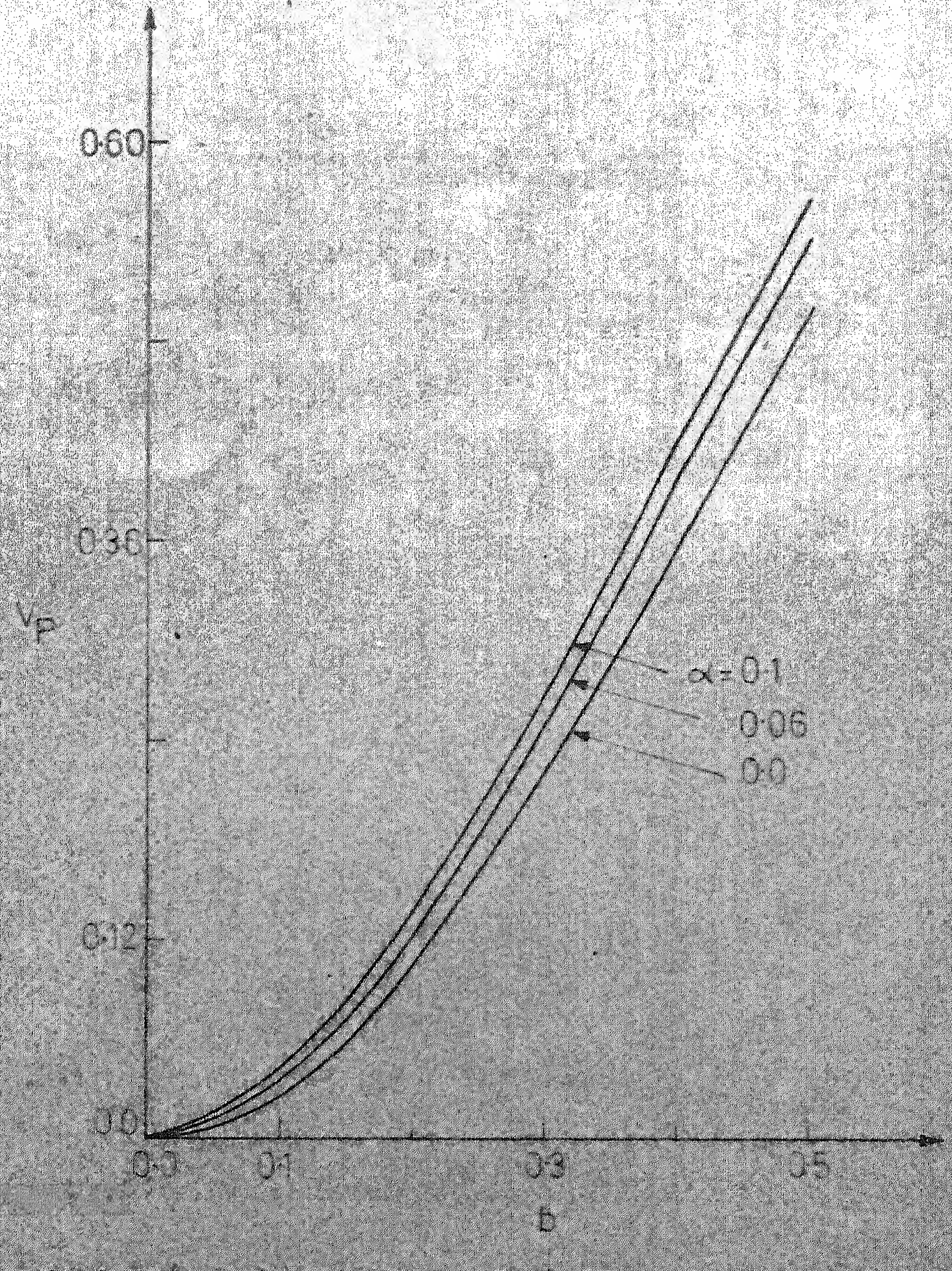

FIG.3·2 VARIATION OF V_P WITH RESPECT TO b

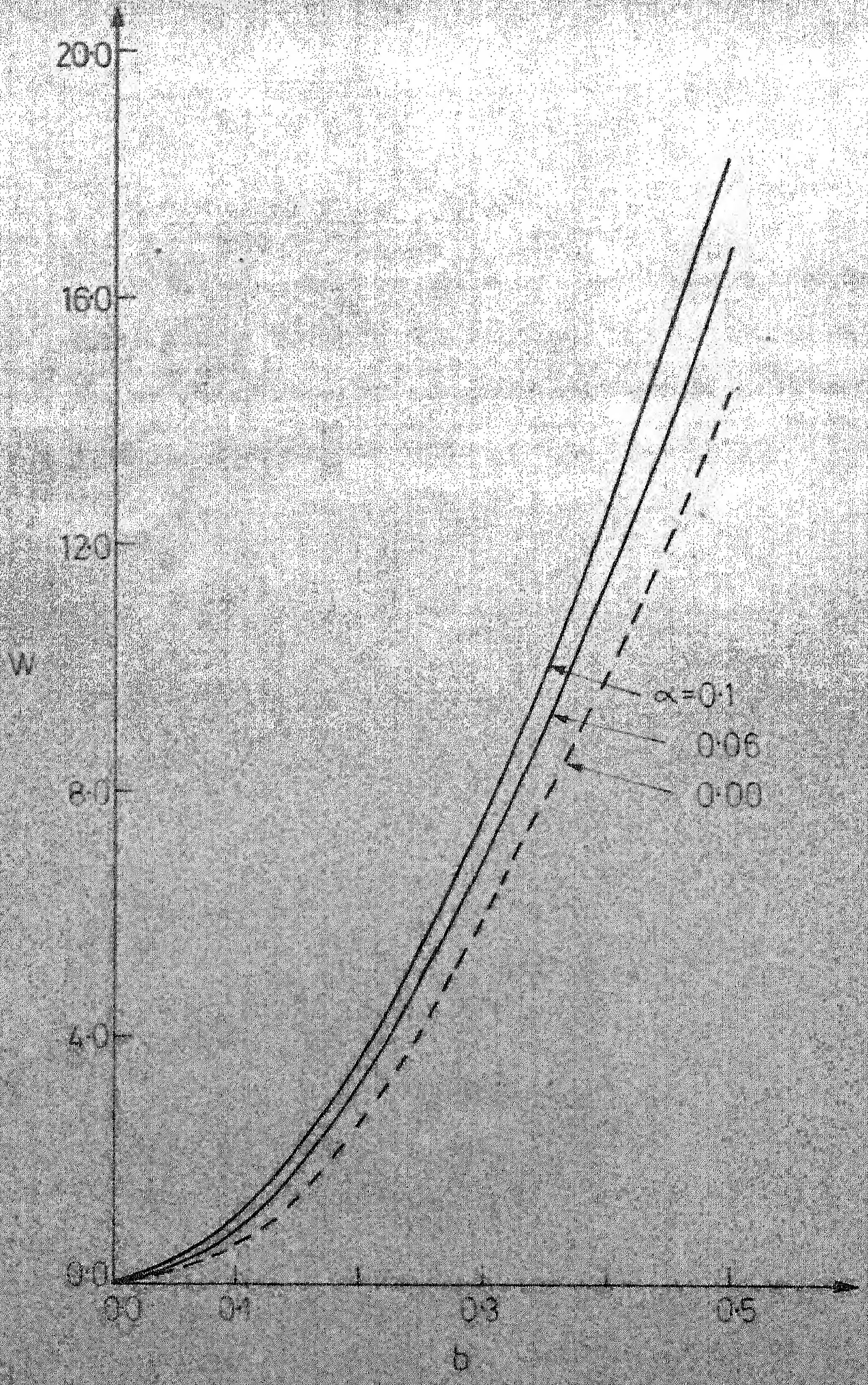

FIG. 3.3 VARIATION OF W WITH b

3 4 EFFECT OF PERIPHERAL LAYER

In the previous section we have seen the effects of dynamical interaction at the walls on the motion. In this section, we study the effects of a more viscous peripheral layer on the swimming of the sperm in a channel The differential equation (3 25) for pressure distribution in this case can be rewritten as

$$[-\frac{1}{12}\frac{G_2^{\pm}}{F_2^{\pm}}\frac{dp^{\pm}}{dx}] + \frac{H_2^{\pm}}{F_2^{\pm}} = Q$$

where $F_2^{\pm}$, $G_2^{\pm}$, $H_2^{\pm}$ are given by equation (3 24-a), (3 24-b) and (3 24-c) After substituting the value of $F_2^{\pm}$, $G_2^{\pm}$, $H_2^{\pm}$ in the above equation, we get,

$$\frac{dp^{\pm}}{dx} = \frac{-12 \ Q \ [1 \mp b \sin x + (\bar{\mu} - 1)(1 - h_1)]}{D} + \frac{6 \ (V_p-2)[(1\mp b \sin x)^2+(\bar{\mu}-1)(1-h_1)^2]-12 \ (1-h_1)(\bar{\mu}-1)(h_1\mp b \sin x)}{D} \tag{3 40}$$

where

$$D = [(1 \mp b \sin x)^2 + (\bar{\mu}-1)(1-h_1)]^2 + 4(\bar{\mu}-1)(1-h_1)(1\mp b \sin x)(h_1 \mp b \sin x)^2$$

We assume the peripheral layer to be very small so that $(1-h_1)^2$ and higher powers can be neglected In such a case, we get,

$$\frac{dp^{\pm}}{dx} = \frac{-12 \ \Omega \ [(1 \mp b \sin x) + \alpha_1]}{(1 \mp b \sin x)^3 \ (1 \mp b \sin x + 4 \ \alpha_1)}$$

$$+ \frac{6 \ (V_p - 2) \ (1 \mp b \sin x) - 12 \ \alpha_1}{(1 \mp b \sin x)^2 \ (4\alpha_1 + 1 \mp b \sin x)} \qquad (3\ 41)$$

in which α_1 is given by

$$\alpha_1 = (\bar{\mu} - 1) \ (1 - h_1) \qquad (3\ 42)$$

Since $\bar{\mu} < 1$, we notice that

$$\alpha_1 < 0 \qquad (3\ 43)$$

Integrating the equation (3 41) between $-\pi$ and π and expanding the resultant in powers of α_1 by assuming $|\alpha_1| << 1$, we obtain, after neglecting the small order terms,

$$\Delta p_\lambda = -12 \ Q \ \pi \ [\frac{b^2+2}{(1-b^2)^{5/2}} - \frac{3\alpha_1(3b^2+2)}{(1-b^2)^{7/2}}]$$

$$+ 12 \ V_p \ \pi \ [\frac{1}{(1-b^2)^{3/2}} - \frac{2\alpha_1(b^2+2)}{(1-b^2)^{5/2}}]$$

$$- 12 \ \pi \ [\frac{2}{(1-b^2)^{3/2}} - \frac{3\alpha_1(b^2+2)}{(1-b^2)^{5/2}}] \qquad (3.44)$$

where $\Delta p_\lambda = p(\pi) - p(-\pi)$

Substituting for $[\frac{\partial u_1}{\partial z}]$ from the equations (3 7) and (3 10-a) in the equation (3 29) we get the equation of balance of fluid force as

$$\frac{8\pi \quad V_P}{\sqrt{(1+\alpha_1)^2 - b^2}} = 2\Delta p_\lambda + \int_{-\pi}^{\pi} b \sin x \left[\frac{dp}{dx}\right] dx$$

$$- \alpha_1 \int_{-\pi}^{\pi} \left[\frac{(1+b \sin x)\, dp^+/dx}{\alpha_1 + 1 - b \sin x} + \frac{(1-b \sin x) dp^-/dx}{\alpha_1 + 1 + b \sin x}\right] dx \qquad (3\ 45)$$

Substituting for $\frac{dp^+}{dx}$ and $\frac{dp^-}{dx}$ and integrating and expanding in powers of α_1, neglecting α_1^2 and higher powers, we obtain from equation (3 45),

$$8\pi \quad V_P \left[\frac{1}{(1-b^2)^{1/2}} - \frac{\alpha_1}{(1-b^2)^{3/2}}\right]$$

$$= 2\Delta p_\lambda - 24 \quad Q\,\pi \left[\frac{3b^2}{(1-b^2)^{3/2}} - \frac{\alpha_1(4b^4 + 19b^2 + 2)}{(1-b^2)^{7/2}}\right]$$

$$+ 12 \quad V_P \quad \pi \left[\frac{2b^2}{(1-b^2)^{3/2}} - \frac{\alpha_1(16b^2+2)}{(1-b^2)^{5/2}}\right]$$

$$- 24 \quad \pi \left[\frac{2b^2}{(1-b^2)^{3/2}} - \frac{\alpha_1(13b^2+2)}{(1-b^2)^{5/2}}\right] \qquad (3\ 46)$$

Solving the equations (3.44) and (3 46) for Q and V_P we get,

$$Q = \frac{\Delta p_\lambda \,(1-b^2)^{3/2}}{12 \quad \pi} + \frac{b^2-1}{1+2b^2} - \frac{\Delta p_\lambda\,(6-9b^2)}{24(1-b^2)^{1/2}(1+2b^2)\pi}$$

$$- \frac{\alpha_1(18 - 15b^2)\, b^2}{2(1-b^2)(1+2b^2)^2} \qquad (3\ 47)$$

$$V_P = \frac{\Delta p_\lambda\,(1-b^2)^{1/2}}{4 \quad \pi} + \frac{3b^2}{1+2b^2} + \frac{\Delta p_\lambda}{\pi}\left[\frac{\alpha_1 \; b^2(6-7b^2+4b^4)}{8(1-b^2)^{3/2}(1+2b^2)}\right]$$

$$- \frac{3\alpha_1}{2} \frac{(8 - 8b^2 + 3b^4)b^2}{(1-b^2)^2(1+2b^2)^2} \qquad (3\ 48)$$

When $\alpha_1 = 0$, we notice that

$$Q = \frac{\Delta p_\lambda \; (1-b^2)^{3/2}}{12\pi} + \frac{(b^2-1)}{1+2b^2} \tag{3 49}$$

$$V_P = \frac{\Delta p_\lambda \; (1-b^2)^{1/2}}{4\pi} + \frac{3b^2}{1+2b^2} \tag{3 50}$$

Which are the same as obtained by Shack and Lardner (1974)

In a free channel case, $\Delta p_\lambda = 0$ Since $\alpha_1 \leq 0$, it may be noted that the effect of a more viscous peripheral layer is to increase the velocity V_P by an amount

$$\frac{3\alpha_1 b^2}{2} \; \frac{(8-8b^2+3b^4)}{(1-b^2)^2(1+2b^2)^2} \tag{3 51}$$

For small amplitudes with $\Delta p_\lambda = 0$,

we get

$$V_P = 3b^2 - 12\alpha_1 b^2 \tag{3 52}$$

$$Q = -1 + 3b^2 - 9\alpha_1 b^2 \tag{3 53}$$

It may be further seen here that as the amplitude b increases, the velocity V_P increases The work done by the sheet against viscous forces is given by equation (3.31) which on integration by parts, yields

$$W = \int_{-\pi}^{\pi} b \sin x \left[\frac{dp}{dx}\right] dx$$

Substituting for $[\frac{dp}{dx}]$ from equations (3.41) we get W as a function of Q and V_p Using the expressions for Q and V_p from equations (3 47) and (3 48) and simplifying we get

$$W = \frac{24 \quad \pi b^2}{(1-b^2)^{1/2} (1+2b^2)} + \frac{3\alpha_1 \ \Delta p_\lambda}{\pi} \quad [\frac{2b^2 - 17b^4 + 17b^6 - 8b^8}{(1-b^2)^3 (1+2b^2)}]$$

$$- 36\alpha_1 b^2 \pi [\frac{2 - 4b^2 + b^4}{(1-b^2)^{5/2} (1+2b^2)^2}] \qquad (3\ 54)$$

for small amplitude ($b << 1$), with $\Delta p_\lambda = 0$, we get

$$W = 24 \quad \pi b^2 - 72\alpha_1 \quad \pi b^2 \qquad (3\ 55)$$

It can be noted here that the effect of the peripheral layer is to increase V_p and W since $\alpha_1 \leq 0$ See fig (3 4) and (3 5)

3 5 CONCLUSIONS

A different approach is suggested to account for the quick transport of sperm through cervical mucus by taking into consideration the physical structure of the mucus and the dynamical interaction of the cervical canal Following Shack and Lardner, the expressions for the flux, propulsive velocity and work done are calculated

It has been pointed out that the propelling velocity and the work done by the sperm increase due to the dynamical interaction of the cervix and due to the presence of the peripheral layer This increase is further enhanced by the increase in the amplitude of the wave produced by the sperm

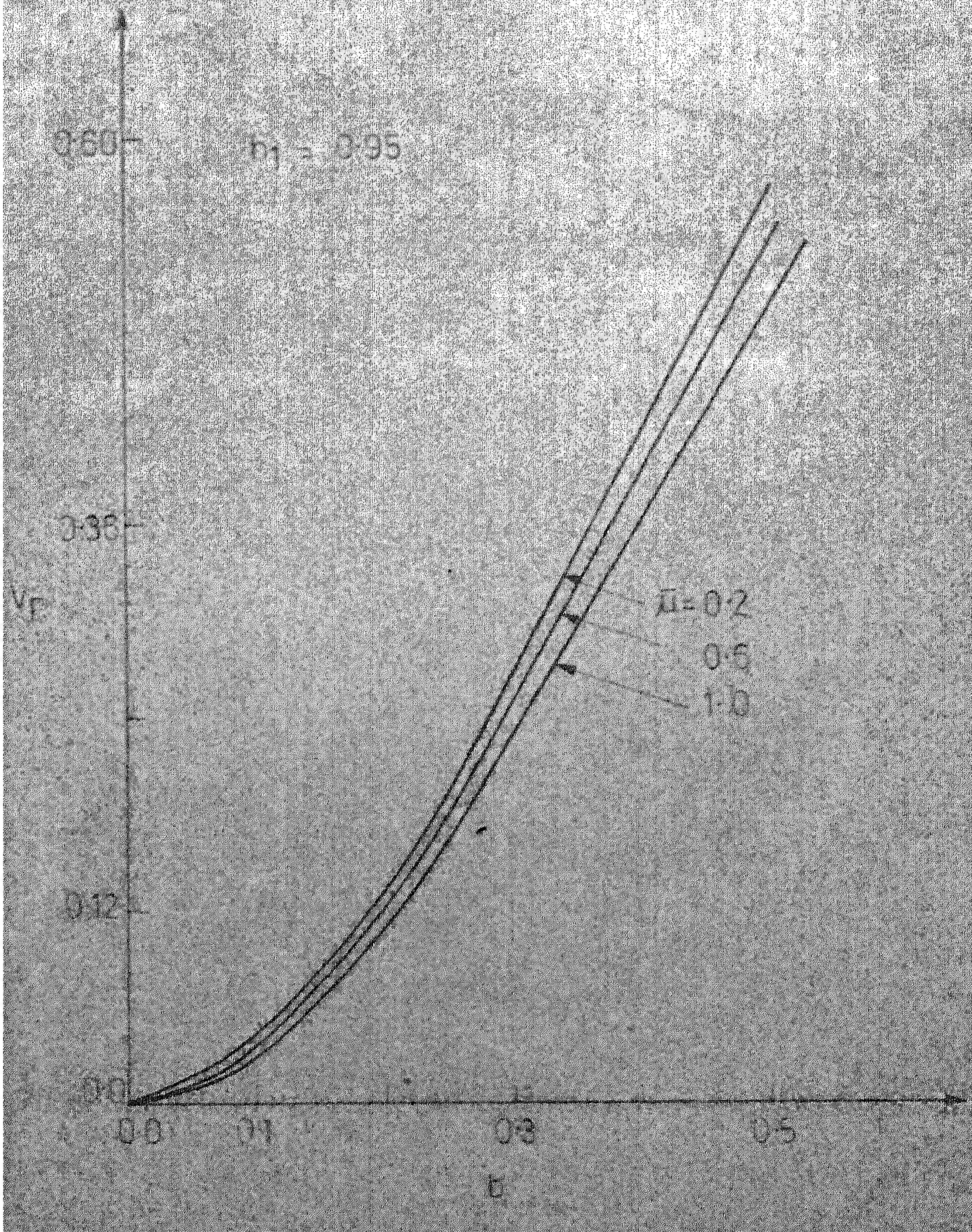

FIG 3·4 VARIATION OF V_P WITH RESPECT To b ·

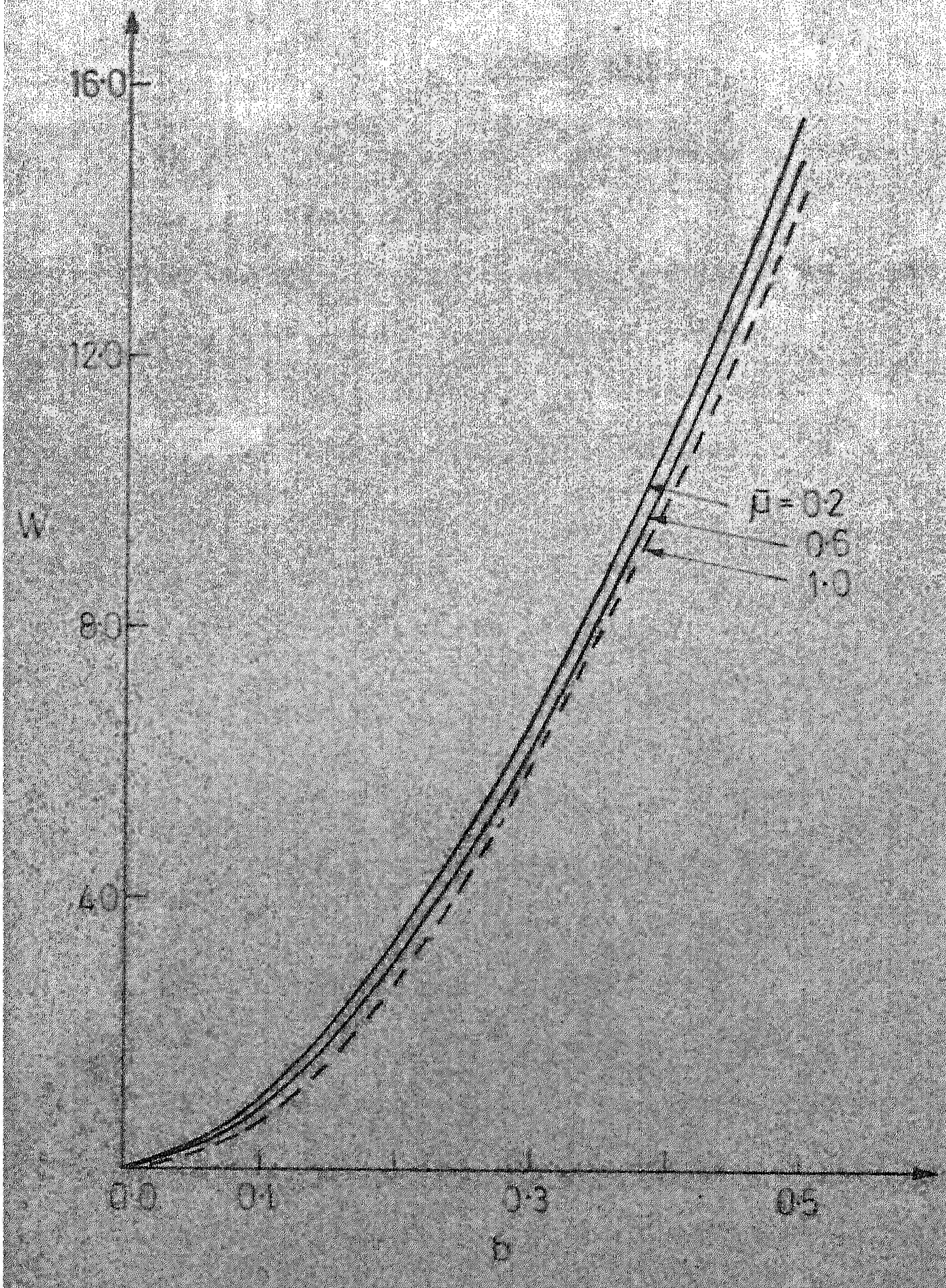

FIG 3·5 VARIATION OF W WITH RESPECT TO b

NOMENCLATURE

b	amplitude of the wave generated by the moving sperm
c	wave velocity
h_o	$h_o = b \sin x$ where $z = b \sin x$ is the equation of the wave in the non-dimensionalised form
h_1	half of width of the central layer
h_2	half of width of the channel
k	wave number
p^+	pressure above the sheet
p^-	pressure below the sheet
Q_x^+	flux in x-direction above the sheet
Q_y^+	flux in y-direction above the sheet
Q_x^-	flux in x-direction below the sheet
Q_y^-	flux in y-direction below the sheet
T^+	fluid force above the sheet
T^-	fluid force below the sheet
(u_1^+, v_1^+, w_1^+)	velocity vector in the central layer above the sheet
(u_2^+, v_2^+, w_2^+)	velocity vector in the peripheral layer above the sheet
(u_1^-, v_1^-, w_1^-)	velocity vector in the central layer below the sheet
(u_2^-, v_2^-, w_2^-)	velocity vector in the peripheral layer below the sheet
V_p	propelling velocity of the sperm
W	work done by the sheet
μ_1	viscosity of the central layer
μ_2	viscosity of the peripheral layer
$\bar{\mu}$	μ_1/μ_2
Δp_λ	pressure difference in one wave length

REFERENCES

1 Brokaw, C J (1972)
Computer simulation of flagellar movement
Biophys J Vol 12, p 584

2 Cameron, A (1966)
The principles of Lubrication
Wiley, New York

3 Davajan et al (1970)
Spermatozoan transport in cervical mucus
Obstetric Gynec Survey Vol 25, p 1

4 ____________ (1971)
A simplified technique for evaluation of biophysical properties of cervical mucus
Amer J Obstet Gynec , Vol 109, p 1042

5 Elstein et al (1971)
Ultra structure of cervical mucus
J Obstet Gynec Brit Comm Vol 78, p 180

6 Hancock, G J (1953)
The self propulsion of microscopic organisms through liquids
Proc Roy Soc , Series A, Vol 217, p 96

7 Lighthill, M J (1974)
Mathematical Bio-fluid dynamics
S I A M

8 Lighthill, M J (1975)
Mathematics of Aquatic Animal Locomotion at Low Reynolds numbers
Winter School in Physiological Fluid Dynamics, I I T ,Delhi, Dec 8 - 18, 1975

9 Moghissi, K S (1969)
Sperm migration in human female genital tract
J Reprod Med , Vol 3, p 73

10 ____________ (1971)
Sperm migration through cervical mucus
In "Pathways to Conception'
(ed A Sherman), Thomas, Springfield

11 Moghissi, K S and Blandau, P J (1972)
Biology of the cervix Summary of a Symposium held at Lake Wilderness, ashington, June 9 - 12, 1971
J Reprod Med , Vol 8, p 21

12 Odeblad, E (1959)
The physics of the cervical mucus
Acta Obstet Gynec Scand , Vol 38, Suppl 1, p 44

13 __________ (1962)
Undulations of macromolecules in cervical mucus
Int J Fert , Vol 7, p 313

14 __________ (1968)
The functional structure of cervical mucus
Acta Obstet Gynec Scand , Vol 47, Suppl 1, p 59

15 __________ (1971)
Cervical factors Nobel symposium 15, Control of Human Fertility (ed Diczfalusy, E and Borell, U) Wiley, New York, p 89

16 Pironneau, O (1975)
The fluid mechanics of uniflagellated micro organisms
Winter school in Physiological Fluid Dynamics, I I T Delhi, Dec 8 - 18, 1975

17 Reynolds, A J (1965)
The swimming of minute organisms
J Fluid Mech , Vol 23, p 241

18 Shack, W J and Lardner, T J (1974)
A long wave length solution for a micro-organism swimming in a channel
Bull Math Biol , Vol 36, p 435

19. Smelser, R E , Shack, W J and Lardner, T J (1974)
The swimming of spermotozoa in a active channel
J Biomech , Vol 7, No 4, p 349.

20 Taylor, Sir Geoffrey (1951)
Analysis of microscopic organisms
Proc Roy Soc , Series A, Vol 209, p 447

CHAPTER IV

EFFECTS OF VISCOSITY VARIATION ON DISPERSION IN A FLUID

4 1 INTRODUCTION

In biological systems such as vascular bed, dispersion plays an important role in the distribution of drug and other nutrients One of the physical parameters to be studied is the coefficient of dispersion which measures the rate of distribution. Various attempts have been made by several authors to investigate the characteristics of dispersion in fluid dynamical situations which can be applied to biological systems

It was Sir Geoffrey Taylor (1953,1954) who first suggested a simplified approach to study the problem of dispersion of a soluble matter in the laminar flow of a Newtonian fluid in circular pipes It was found that, relative to a plane moving with the mean speed of the flow, the solute appears to diffuse with an apparent diffusion coefficient $\frac{R^2 U^2}{48D}$ where U is the average speed of the flow, D is the molecular diffusion coefficient and R is the radius of the pipe. Taylor imposed certain restrictions on his analysis which were later removed by Aris (1956) Numerous authors utilised Taylor's approach to find concentration profiles in the case of non-Newtonian fluids Fan and Hwang (1965) considered powerlaw fluid while Fan and Wang (1966) investigated the dispersion in Bingham plastic and Ellis model fluids.

The case of Reiner-Philippoff model fluid was considered by Ghoshal (1971) Shah and Cox (1974) studied the Eyring model Crank (1968, 1975), Crank et al (1972), Cumming et al (1966), Gill and Sankarasubramanian (1970, 1971) have also studied in detail the steady and unsteady state diffusion in various cases

None of these studies has taken into consideration the homogeneous and heterogeneous reactions of the solute with the solvent Recently Gupta and Gupta (1972) discussed "the effects of homogeneous and heterogeneous reactions on the dispersion of a solute in the laminar flow between two plates"

In all these investigations the viscosity of the solvent is taken to be constant. However, for many biological fluids and fluids of suspensions such as blood, the viscosity varies across the tube, Haynes (1960), Haynes et al. (1959), Lih (1969, 1975), Whitmore (1968), Middleman (1972) It is therefore desirable to study the effects of viscosity variation on the dispersion of a solute in a flowing medium.

In this Chapter, we study the effects of viscosity variation on the dispersion of a solute in the laminar flow between two plates and circular pipes by taking into consideration the homogeneous and heterogeneous reactions of the solvent with the solute

4 2 DISPERSION THROUGH A FLUID FLOW BETWEEN TWO PLATES, WITH HOMOGENEOUS REACTION

Consider the laminar flow of a viscous liquid of varying viscosity under a constant pressure gradient between two parallel plates

distant 2h apart The physical configuration and the coordinate system are shown in the Fig (4 1) The viscosity variation is taken as a known function

$$\mu = \mu(y) \qquad 0 \leq y \leq h$$

and is assumed to be symmetrical about $y = 0$ i e

$$\mu(-y) = \mu(y) \tag{4 1}$$

The fully developed velocity profile in this case is given by

$$u = \left(-\frac{dP}{dx}\right) \int_y^h \frac{y}{\mu}\, dy \tag{4 2}$$

The average velocity, which is defined as

$$\bar{u} = \frac{1}{2h} \int_{-h}^{h} u\, dy \tag{4 3}$$

is given by

$$\bar{u} = \left(-\frac{dP}{dx}\right) \frac{1}{h} \int_0^h \frac{y^2}{\mu}\, dy \tag{4 4}$$

We assume that the diffusing matter while dispersing, undergoes a first order irreversible chemical raction in the fluid under isothermal conditions In such a case, the equation for concentration c is given by

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial y^2} - K\, c \tag{4 5}$$

where D is the molecular diffusion coefficient. K is the reaction rate constant. It may be noted that, in writing the equation (4 5)

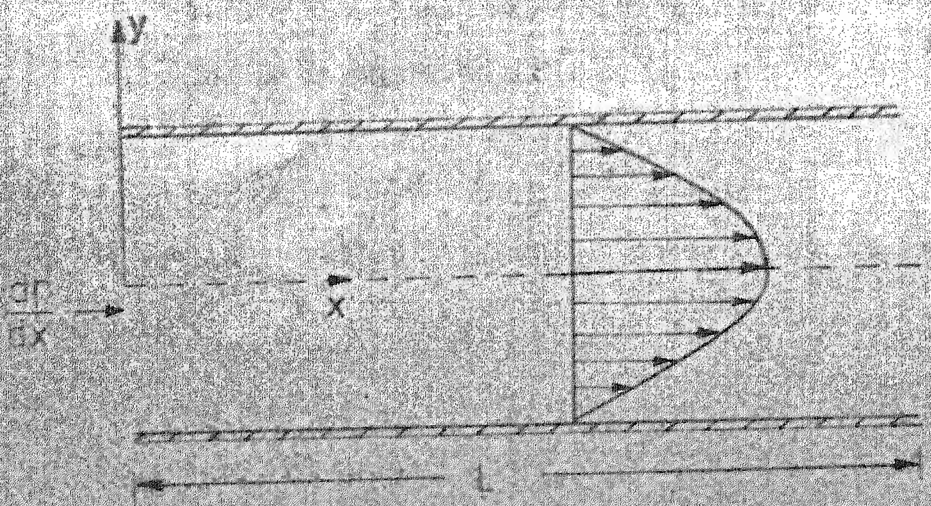

FIG.4·1 PARALLEL PLATE GEOMETRY FOR DISPERSION IN A FLUID OF VARIABLE VISCOSITY

we have assumed that

$$\frac{\partial^2 c}{\partial x^2} \ll \frac{\partial^2 c}{\partial y^2}$$

Relative to a plane moving with the mean speed $\bar{u}$ of the flow, the velocity of the fluid is given by

$$u_x = u - \bar{u} = \left(-\frac{dP}{dx}\right) h^2 \, [G_o - f(\eta)] \tag{4 6}$$

where

$$G_o = \int_0^1 \frac{t-t^2}{\mu(t)} \; dt \tag{4 7}$$

$$f(\eta) = \int_0^\eta \frac{t}{\mu(t)} \; dt \tag{4 8}$$

Using equation (4 6), the concentration relative to the moving plane is given by

$$\frac{\partial c}{\partial t} + u_x \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial y^2} - K c \tag{4 9}$$

Non-dimensionalising the variables by

$$\theta = \frac{t}{\bar{t}} , \; \bar{t} = \frac{L}{\bar{u}} , \; \xi = \frac{x-\bar{u}\,t}{L} , \; \eta = \frac{y}{h}$$

where L is a characteristic length along x-axis, the diffusion equation (4 9) can be written as

$$\frac{1}{\bar{t}} \frac{\partial c}{\partial \theta} + \frac{u_x}{L} \frac{\partial c}{\partial \xi} = \frac{D}{h^2} \frac{\partial^2 c}{\partial \eta^2} - K c \tag{4 10}$$

Following Taylor, we now assume that partial equilibrium is established in any cross-section of the channel so that the diffusion equation can be written as

$$\frac{\partial^2 c}{\partial \eta^2} - \alpha^2 c = H \; [G_o - f(\eta)] \tag{4 11}$$

where

$$\alpha^2 = \frac{Kh^2}{D} \tag{4 12}$$

$$H = \frac{h^4}{DL^2} \left(\frac{\partial c}{\partial \xi}\right)\left(- \frac{dP}{d\xi}\right) \tag{4 13}$$

and $\left(\frac{\partial c}{\partial \xi}\right)$ is a constant

The boundary conditions are

$$\frac{\partial c}{\partial \eta} = 0 \quad \text{at} \quad \eta = \pm 1 \tag{4 14}$$

Solving the equation (4 11) with the above boundary condition, we get

$$c = A \cosh \alpha\eta - \frac{H \sinh \alpha\eta}{\alpha} \int_0^\eta \cosh \alpha t \quad f(t) \, dt$$

$$+ \frac{H \cosh \alpha\eta}{\alpha} \int_0^\eta \sinh(\alpha t) f(t) \, dt - \frac{HG_o}{\alpha^2} \tag{4 15}$$

where

$$A = \frac{H}{\alpha \sinh \alpha} \int_0^1 \cosh (\alpha - \alpha t) \quad f(t) \quad dt \tag{4 16}$$

The volumetric rate at which the solute is transported across a section of the channel of unit breadth is

$$Q = \int_{-h}^{h} c \; u_x \; dy \tag{4.17}$$

Using equations (4 6) and (4 15), we obtain

$$Q = -\frac{2h^7 \left(\frac{dP}{dx}\right)^2 \left(\frac{\partial c}{\partial x}\right)}{D} \; F(\alpha) \tag{4 18}$$

where

$$F(\alpha) = \frac{1}{\alpha} \left(F_1 - F_2 + F_3\right)$$

$$F_1 = \left[\int_0^1 \frac{\cosh(\alpha - \alpha t)}{\sinh \alpha} \; f(t)dt\right] \int_0^1 \cosh(\alpha t)\left[f(t) - G_0\right] \, dt$$

$$F_2 = \int_0^1 \left\{\sinh \alpha t \quad [f(t) - G_0] \quad \left[\int_0^t \cosh \alpha s \; f(s) \, ds\right]\right\}dt$$

and,

$$F_3 = \int_0^1 \left\{\cosh(\alpha t) \; [f(t) - G_0] \quad \left[\int_0^t f(s) \sinh \alpha s \, ds\right]\right\}dt \tag{4 19}$$

Comparing with Fick's law of diffusion the effective dispersion coefficient D* is given by

$$D^* = \frac{h^6 \left(\frac{dP}{dx}\right)^2}{D} \; F(\alpha) \tag{4 20}$$

It is noted that this expression is applicable for any given general function $\mu(y)$ However, to study specifically the effects of viscosity variation on dispersion, the following relation is assumed which may arise in physiological situations.

$$\mu(y) = \mu_o \, e^{-\delta y} \qquad (\delta << 1) \qquad (4\ 21\text{-a})$$

or

$$\frac{1}{\mu(y)} = \frac{1}{\mu_o} \, (1 + \delta y) \qquad (4\ 21\text{-b})$$

Using the equations (4 8), (4 21-b) in equation (4 15) we obtain the concentration profile as follows

$$c = M_1 \cosh \alpha\eta + M_2 \left(\frac{1}{3} - \frac{2}{\alpha^2} - \eta^2\right)$$
$$+ \delta \left[N_1 \cosh \alpha\eta + N_2 \sinh \alpha\eta + M_2\right] \qquad (4\ 22)$$

where

$$M_1 = \frac{h^4}{DL^2 \mu_o \alpha^3 \sinh \alpha} \left(\frac{\partial c}{\partial \xi}\right)\left(\frac{\partial P}{\partial \xi}\right) \qquad (4\ 23\text{-a})$$

$$M_2 = \frac{h^4}{2DL^2 \mu_o \alpha^2} \left(\frac{\partial c}{\partial \xi}\right)\left(\frac{\partial P}{\partial \xi}\right) \qquad (4\ 23\text{-b})$$

$$N_1 = \frac{h^4}{DL^2 \mu_o \sinh \alpha} \left(\frac{\partial c}{\partial \xi}\right)\left(\frac{\partial P}{\partial \xi}\right) \left[\frac{1}{\alpha^3} + \frac{2}{\alpha^5} - \frac{2 \cosh \alpha}{\alpha^5}\right] \qquad (4\ 23\text{-c})$$

and

$$N_2 = \frac{2h^4}{DL^2 \mu_o \alpha^5} \left(\frac{\partial c}{\partial \xi}\right)\left(\frac{\partial P}{\partial \xi}\right) \qquad (4\ 23\text{-d})$$

The flux Q and the effective dispersion coefficient D* are given by equations (4 18) and (4 20) in which $F(\alpha)$ takes the form

$$F(\alpha,\delta) = \frac{1}{\mu_o^2 \alpha^2}\left[\left(\frac{1}{45} - \frac{1}{3\alpha^2} + \frac{\coth \alpha}{\alpha^3} - \frac{1}{\alpha^4}\right) + \delta\left(\frac{1}{36} - \frac{1}{2\alpha^2} - \frac{4}{\alpha^4} + \frac{2 \coth \alpha}{\alpha^3} + \frac{4 \coth \alpha}{\alpha^5} - \frac{4 \operatorname{cosech} \alpha}{\alpha^5}\right)\right] \qquad (4\ 24)$$

$$= \frac{1}{\mu_o^2} \bar{F}(\alpha,\delta)$$

To see the effects of α analytically on dispersion coefficient, we expand $F(\alpha,\delta)$ for small α as follows

$$\mu_0^2 F(\alpha,\delta) = \left(\frac{2}{945} - \frac{\alpha^2}{4725}\right) + \delta\left(\frac{11}{4320} - \frac{229\alpha^2}{945 \times 960}\right) \tag{4 25}$$

As $\alpha \to 0$,

$$\underset{\alpha \to 0}{\mathrm{Lt}}\, \mu_0^2 F(\alpha,\delta) = \frac{2}{945} + \frac{11\delta}{4320} \tag{4 26}$$

which generalises the result of Wooding (1960) for a fluid of variable viscosity

When $\delta = 0$, we get the case of a fluid with constant viscosity for which we have

$$F(\alpha) = \frac{1}{\mu_0^2 \alpha^2}\left[\frac{1}{45} - \frac{1}{3\alpha^2} + \frac{\coth\alpha}{\alpha^3} - \frac{1}{\alpha^4}\right]$$

This agrees with the result of Gupta et al (1972) For a fixed δ we notice that $F(\alpha,\delta)$ decreases as α increases as observed by Gupta et al (1972) This is due to the presence of homogeneous reaction of the solute with the fluid. But it can be seen from equation (4 25) that as δ increases $F(\alpha,\delta)$ increases a fixed α, which implies that the equivalent coefficient of dispersion increases as the viscosity towards the wall decreases

To see the effects of α, in general, $\bar{F}(\alpha,\delta)$, given by equation (4 24) has been calculated and tabulated in table (4 1).

<u>Table 4 1</u>

α	δ	$\overline{F}(\alpha, \delta)$
	0 00	$0\ 15133070 \times 10^{-2}$
	0 01	$0\ 15315713 \times 10^{-2}$
2 0	0 1	$0\ 16959501 \times 10^{-2}$
	0 2	$0\ 18785933 \times 10^{-2}$
	0 3	$0\ 20612364 \times 10^{-2}$
	0 4	$0\ 22438796 \times 10^{-2}$
	0.00	$0\ 81988283 \times 10^{-3}$
	0 01	$0\ 82984501 \times 10^{-3}$
4 0	0.1	$0\ 91950461 \times 10^{-3}$
	0 2	$0\ 10191264 \times 10^{-2}$
	0 3	$0.11187481 \times 10^{-2}$
	0 4	$0\ 12183699 \times 10^{-2}$
	0 00	$0\ 46725123 \times 10^{-3}$
	0 01	$0\ 47296614 \times 10^{-3}$
6.0	0 1	$0\ 52440040 \times 10^{-3}$
	0.2	$0\ 58154956 \times 10^{-3}$
	0 3	$0\ 63869873 \times 10^{-3}$
	0 4	$0\ 69584790 \times 10^{-3}$

α	δ	$\bar{F}(\alpha,\delta)$
	0 00	$0\ 29254489 \times 10^{-3}$
	0 01	$0\ 29614129 \times 10^{-3}$
	0 1	$0\ 32850888 \times 10^{-3}$
8 0	0 2	$0\ 36447287 \times 10^{-3}$
	0 3	$0\ 40043686 \times 10^{-3}$
	0 4	$0\ 43640085 \times 10^{-3}$
	0 00	$0\ 19788888 \times 10^{-3}$
	0.01	$0\ 20033066 \times 10^{-3}$
	0 1	$0\ 22230665 \times 10^{-3}$
10 0	0 2	$0\ 24672443 \times 10^{-3}$
	0 3	$0\ 27114220 \times 10^{-3}$
	0 4	$0\ 29555997 \times 10^{-3}$

It can be seen from table (4 1) that as α increases $\overline{F}(\alpha,\delta)$ decreases for a fixed δ. These results are same as obtained analytically

4 3 EFFECTS OF COMBINED HOMOGENEOUS AND HETEROGENEOUS REACTION

Considering the effects of chemical reaction taking place in the fluid and at the walls the diffusion equation after non-dimensionalising the variables and following Taylor (1953), becomes

$$\frac{\partial^2 c}{\partial \eta^2} - \alpha^2 c = H \; [G_o - f(\eta)] \tag{4 27}$$

where H, G_o, $f(\eta)$ are given by equations (4 13), (4 7) and (4 8) respectively

The boundary conditions are

$$\frac{\partial c}{\partial \eta} \pm \beta c = 0 \quad \text{at} \quad \eta = \pm 1 \tag{4 28}$$

where

$$\beta = sh$$

and s is the heterogeneous reaction rate constant

Solving equation (4 27) with boundary conditions(4 28) we obtain

$$c = A \cosh \alpha\eta - \frac{H}{\alpha}\int_0^\eta f(t) \quad \sinh \alpha(\eta - t)\, dt - \frac{H\, G_o}{\alpha^2} \tag{4 29}$$

where

$$A = \frac{H\,[S_1 + \beta(\frac{S_2}{\alpha} + \frac{G_o}{\alpha^2})]}{\alpha \sinh \alpha + \beta \cosh \alpha} \tag{4 30}$$

$$S_1 = \int_0^1 f(t) \quad \cosh(\alpha - \alpha t)\, dt$$

and

$$S_2 = \int_0^1 f(t) \quad \sinh(\alpha - \alpha t)\, dt$$

The volumetric rate at which the solute is transported across a section of the channel of unit breadth is given by equation (4 17) Substituting for c and u_x from equations (4 29) and (4 6) we get

$$Q = \frac{-2h^7 \left(\frac{dP}{dx}\right)^2 \left(\frac{\partial c}{\partial x}\right)}{D} G(\alpha,\beta) \tag{4 31}$$

where

$$G(\alpha,\beta) = G_1(\alpha,\beta) + G_2(\alpha) \tag{4 32}$$

$$G_1(\alpha,\beta) = \left[\frac{S_1 + \beta\left(\frac{S_2}{\alpha} + \frac{G_o}{\alpha^2}\right)}{\alpha\sinh\alpha + \beta\cosh\alpha}\right] \int_0^1 [f(t) - G_o] \cosh\alpha t \quad dt$$

and

$$G_2(\alpha) = \frac{1}{\alpha} \int_0^1 \left\{ [f(t) - G_o] \left[\int_0^t f(s) \sinh\alpha(s-t)\, ds \right] \right\} dt$$

Comparing with Fick's law of diffusion, the effective dispersion coefficient is given by

$$D^* = \frac{h^6 (dP/dx)^2}{D} G(\alpha,\beta) \tag{4 33}$$

where $G(\alpha,\beta)$ is given by equation (4 32)

As in the last section, we assume

$$\mu(y) = e^{-\delta y} \quad (\delta \ll 1)$$

so that

$$\frac{1}{\mu(y)} = \frac{1}{\mu_o} \; [1 + \delta y]$$

For this function of μ, the effective dispersion coefficient is calculated as

$$D^* = \frac{h^6 (dP/dx)^2}{D} \, G^*(\alpha, \beta, \delta) \qquad (4\ 34)$$

where

$$G^*(\alpha, \beta, \delta) = \frac{A^* + \delta B^*}{\mu_o^2 \; (\alpha \sinh \alpha + \beta \cosh \alpha)} = \frac{1}{\mu_o^2} \bar{G}^* (\alpha,\beta,\delta) \qquad (4\ 35)$$

$$A^* = \frac{1}{45\alpha^7} [\{ \alpha^6 - (\alpha^2+3)(15\alpha^2 + 5\alpha^2\beta + 15\beta)\} \quad \sinh \alpha$$

$$+ (45\alpha^2 + 15\alpha^2 \beta + 45\beta + \alpha^4 \beta) \quad \alpha \cosh \alpha] \qquad (4\ 36)$$

and

$$B^* = \frac{-4}{\alpha^6} - 4\beta \left(\frac{1}{3\alpha^6} + \frac{1}{\alpha^8}\right) + \sinh \alpha \quad [\left(\frac{1}{36\alpha} - \frac{1}{2\alpha^3} - \frac{4}{\alpha^5}\right)$$

$$- \beta \left(\frac{1}{6\alpha^3} + \frac{11}{6\alpha^5} + \frac{4}{\alpha^7}\right)] + \cosh \alpha \quad [\left(\frac{2}{\alpha^4} + \frac{4}{\alpha^6}\right)$$

$$+ \beta\left(\frac{1}{36\alpha^2} + \frac{2}{3\alpha^4} + \frac{10}{3\alpha^6} + \frac{4}{\alpha^8}\right)] \qquad (4.37)$$

When $\delta = 0$, $G^*(\alpha, \beta, \delta)$ reduces to

$$G_1^*(\alpha, \beta) = \frac{1}{\alpha^2} \; (A_1^* \sinh \alpha + B_1^* \cosh \alpha) \qquad (4\ 38)$$

where

$$A_1^* = \frac{\alpha^6 - (\alpha^2+3)(15\alpha^2 + 5\alpha^2\beta + 15\beta)}{45\mu_o^2\,\alpha^5(\alpha\sinh\alpha + \beta\cosh\alpha)} \tag{4 39}$$

$$B_1^* = \frac{45\alpha^2 + 15\alpha^2\beta + 45\beta + \alpha^4\beta}{45\mu_o^2\,\alpha^4(\alpha\sinh\alpha + \beta\cosh\alpha)} \tag{4 40}$$

which can be seen to agree with Gupta et al (1972)

As $\beta \to 0$, we get

$$\underset{\beta\to 0}{\mathrm{Lt}}\; G^*(\alpha,\beta,\delta) = \frac{1}{\alpha^2}\Big[\Big(\frac{1}{45} - \frac{1}{3\alpha^2} - \frac{1}{\alpha^4} + \frac{\coth\alpha}{\alpha^3}\Big)$$

$$+\,\delta\Big(\frac{1}{36} - \frac{1}{2\alpha^2} - \frac{4}{\alpha^4} - \frac{4\ \mathrm{cosech}\,\alpha}{\alpha^5}$$

$$+\,\frac{2\coth\alpha}{\alpha^3} + \frac{4\coth\alpha}{\alpha^5}\Big)\Big] \tag{4 41}$$

which agrees with equation (4 24)

To see the effects of β and δ on the effective dispersion coefficient, the function $\overline{G^*}(\alpha,\beta,\delta)$ given by equation (4 35) is calculated for $\alpha = 1\ 0$ and tabulated in table (4 2)

It can be seen from table (4.2) that, for a given α, $\overline{G^*}(\alpha,\beta,\delta)$ decreases as β increases whereas it increases with increasing δ when β is held fixed

Table 4 2 $(\alpha = 1\ 0)$

β	δ	$\overline{G}^*(\alpha,\beta,\delta)$
2 0	0 00	$0\ 16969080 \times 10^{-2}$
	0 01	$0\ 17172264 \times 10^{-2}$
	0 1	$0\ 19000921 \times 10^{-2}$
	0 2	$0\ 21032761 \times 10^{-2}$
	0 3	$0\ 23064602 \times 10^{-2}$
	0 4	$0\ 25096442 \times 10^{-2}$
4 0	0 0	$0\ 16605727 \times 10^{-2}$
	0 01	$0\ 16804359 \times 10^{-2}$
	0 1	$0\ 18592050 \times 10^{-2}$
	0 2	$0\ 20578372 \times 10^{-2}$
	0 3	$0\ 22564695 \times 10^{-2}$
	0 4	$0\ 24551018 \times 10^{-2}$
6 0	0 00	$0\ 16457162 \times 10^{-2}$
	0 01	$0\ 18053926 \times 10^{-2}$
	0 1	$0\ 18424803 \times 10^{-2}$
	0 2	$0\ 20392444 \times 10^{-2}$
	0 3	$0\ 22360085 \times 10^{-2}$
	0 4	$0\ 24327726 \times 10^{-2}$

8.0	0.00	$0.16376549 \times 10^{-2}$
	0.01	$0.16572301 \times 10^{-2}$
	0.1	$0.18334072 \times 10^{-2}$
	0.2	$0.20291595 \times 10^{-2}$
	0.3	$0.22249119 \times 10^{-2}$
	0.4	$0.24206642 \times 10^{-2}$
10.00	0.00	$0.16325898 \times 10^{-2}$
	0.01	$0.16521009 \times 10^{-2}$
	0.1	$0.18277007 \times 10^{-2}$
	0.2	$0.20228116 \times 10^{-2}$
	0.3	$0.22179225 \times 10^{-2}$
	0.4	$0.24130334 \times 10^{-2}$

Thus it may be concluded that the effective dispersion coefficient decreases due to homogeneous and heterogeneous reactions and it increases as the viscosity of the fluid decreases towards the wall

4.4 EFFECTS OF HOMOGENEOUS REACTION ON DISPERSION IN A FLUID FLOWING IN A CIRCULAR TUBE

In the last two sections we have studied the effects of homogeneous and heterogeneous reactions on the dispersion of a solute in a fluid flowing between two parallel plates Corresponding results will be obtained in this section when the fluid is flowing in a circular tube.

Consider the laminar flow of a viscous liquid of varying viscosity under constant pressure gradient in a circular pipe whose physical configuration is shown in Fig (4 2) As before we consider the viscosity variation as a known function,

$$\mu = \mu(r) \quad (0 \leq r \leq R) \tag{4 43}$$

The fully developed velocity profile can be obtained in this case as

$$W = -\frac{1}{2}\left(\frac{dp}{dz}\right) \int_r^R \frac{r}{\mu}\, dr \tag{4 44}$$

The average velocity, which is defined as

$$\bar{W} = \frac{2}{R^2} \int_0^R rW dr \tag{4 45}$$

is given by

$$\bar{W} = \left(\frac{-1}{2R^2} \frac{dp}{dz}\right) \int_0^R \frac{r^3 dr}{\mu} \tag{4 46}$$

We assume that the solute diffuses while undergoing irreversible reaction in the liquid under isothermal conditions The equation for concentration c is given by

$$\frac{D}{r} \frac{\partial}{\partial r}\left(r \frac{\partial c}{\partial r}\right) - Kc = W \frac{\partial c}{\partial z} + \frac{\partial c}{\partial t} \tag{4.47}$$

Relative to a plane moving with the mean speed of the flow, the fluid velocity is given by

$$W_x = W - \bar{W} = \left(-\frac{1}{2} \frac{dp}{dz} R^2\right) \left[G_o - f(n)\right] \tag{4 48}$$

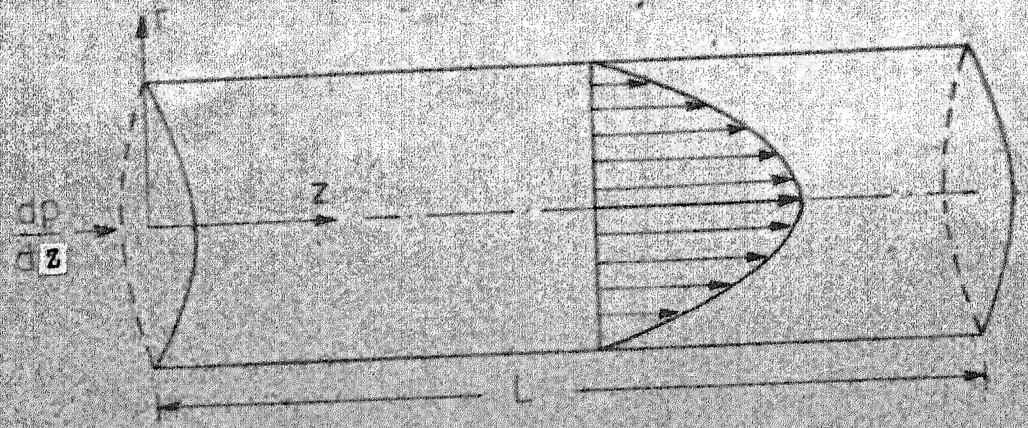

FIG. 4.2 CIRCULAR TUBE GEOMETRY FOR DISPERSION IN A FLUID OF VARIABLE VISCOSITY

where G_o and $f(\eta)$ are given by equations (4 7) and (4 8) Nondimensionalising the variables by

$$\theta = \frac{t}{\bar{t}}\ ,\ \bar{t} = \frac{L}{W}\ ,\ \xi = \frac{z-\bar{W}t}{L}\ ,\ \eta = \frac{r}{R}$$

and following the Taylor's approach equation (4 47) can be written as,

$$\frac{1}{\eta}\ \frac{\partial}{\partial\eta}\ (\eta\ \frac{\partial c}{\partial\eta}\ ) - \alpha^2 c = H\ [G_o - f(\eta)] \tag{4 49}$$

where

$$\alpha^2 = \frac{KR^2}{D} \tag{4 50-a}$$

$$H = -\frac{1}{2}\ (\frac{dp}{dz})\ (\frac{\partial c}{\partial\xi})\ \frac{R^4}{DL} \tag{4 50-b}$$

The boundary conditions are

$$\frac{\partial c}{\partial\eta} = 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \eta = 1 \tag{4 51}$$

Solving equation (4 49) with the above boundary condition, we get

$$c = [\frac{H\ S}{I_1(\alpha)}]\ I_o\ (\alpha\eta) - \frac{H\ G_o}{\alpha^2} - H\ F(\eta) \tag{4 52}$$

where

$$S = \int_0^1 t\ [I_o(\alpha t)\ K_1(\alpha) + I_1(\alpha)\ K_o(\alpha t)]\ f(t)dt \tag{4 53}$$

$$F(\eta) = \int_0^\eta t\ [I_o(\alpha\eta)\ K_o(\alpha t) - I_o(\alpha t)\ K_o(\alpha\eta)]\ f(t)dt \tag{4 54}$$

I_o, K_o are zero order modified Bessel functions and I_1, K_1 are of order one of first and second kind respectively The

volumetric rate at which the solute is transported across a section of the tube is

$$Q = 2\pi \int_{o}^{R} r\, c\, W_x\, dr \qquad (4\ 55)$$

Using equations (4 48) and (4 52) we get

$$Q = -\frac{\pi R^8}{2D} \left(\frac{dp}{dz}\right)^2 \left(\frac{\partial c}{\partial z}\right) X(\alpha) \qquad (4\ 56)$$

where

$$X(\alpha) = \int_{o}^{1} \eta \left[F(\eta) - \frac{S\, I_o(\alpha\eta)}{I_1(\alpha)}\right] \left[G_o - f(\eta)\right] d\eta \qquad (4\ 57)$$

Comparing with Fick's law of diffusion the effective dispersion coefficient is obtained as

$$D^* = \frac{1}{2} \left(\frac{dp}{dz}\right)^2 \frac{R^6}{D}\, X(\alpha) \qquad (4\ 58)$$

where $X(\alpha)$ is given by equation (4 57)

If the fluid has a constant viscosity, expressions for concentration and equivalent dispersion coefficient can be obtained, after some manipulation with Bessel functions, from the equations (4 52) and (4 58), as

$$c = \frac{-H}{\mu_o\, \alpha^3\, I_1(\alpha)}\, I_o(\alpha\eta) - \frac{H}{2\mu_o\, \alpha^2} \left[\frac{1}{2} - \frac{4}{\alpha^2} - \eta^2\right] \qquad (4\ 59)$$

and

$$D^* = \frac{R^2 \bar{W}^2}{D\alpha^4} \left[\frac{32\, I_2(\alpha)}{\alpha\, I_1(\alpha)} + \frac{\alpha^2}{3} - 8\right] \qquad (4\ 60)$$

when α is small, D^* can be written as

$$D^* = \frac{R^2\, \bar{W}^2}{48\, D} \left(1 - \frac{\alpha^2}{15}\right) \qquad (4\ 61)$$

We observe when $\alpha = 0$, the Taylor's result is obtained We notice also as α increases D^* decreases

It may be noted that the expression for D^* given by equation (4 58) is for any known function of viscosity variation denoted by $\mu(r)$ However, to study specifically the effects of viscosity variation on dispersion the following relation is assumed as before ,

$$\mu(r) = \mu_o \bar{e}^{\delta r} \quad (\delta << 1) \tag{4 62}$$

or

$$\frac{1}{\mu(r)} = \frac{1}{\mu_o}(1 + \delta r) \tag{4 63}$$

The effective dispersion coefficient is calculated from equation (4 58) using the equation (4 63) for viscosity variation.

To see the effects of α and δ on the effective dispersion coefficient, polynomial expansions of the Bessel functions have been used to evaluate the integrals in $X(\alpha, \delta)$ by neglecting α^4 and higher powers The values of $\mu_o^2 X(\alpha, \delta)$ have then been calculated and tabulated in table (4 3)

Table 4 3

α	δ	$\mu_0^2\gamma(\alpha,\delta)$
	0 00	0 65060771 $\times 10^{-3}$
	0 01	0 65924755 $\times 10^{-3}$
0 1	0 1	0 74129019 $\times 10^{-3}$
	0 2	0 84149353 $\times 10^{-3}$
	0 3	0 95121762 $\times 10^{-3}$
	0 4	0 10704625 $\times 10^{-2}$
	0 00	0 64930735 $\times 10^{-3}$
	0 01	0 65793009 $\times 10^{-3}$
0.2	0 1	0 73981530 $\times 10^{-3}$
	0 2	0.83983510 $\times 10^{-3}$
	0 3	0 94936667 $\times 10^{-3}$
	0 4	0 10684100 $\times 10^{-2}$
	0.00	0.64714460 $\times 10^{-3}$
	0 01	0 65573898 $\times 10^{-3}$
0.3	0 1	0 73736230 $\times 10^{-3}$
	0.2	0 83707686 $\times 10^{-3}$
	0 3	0 94628819 $\times 10^{-3}$
	0 4	0 10649963 $\times 10^{-2}$

α	δ	$\mu_o^2 X(\alpha,\delta)$
0.4	0.00	$0.54412641 \times 10^{-3}$
	0.01	$0.65268124 \times 10^{-3}$
	0.1	$0.73393912 \times 10^{-3}$
	0.2	$0.83322773 \times 10^{-3}$
	0.3	$0.94199227 \times 10^{-3}$
	0.4	$0.10602327 \times 10^{-2}$
0.5	0.00	$0.64026262 \times 10^{-3}$
	0.01	$0.64876683 \times 10^{-3}$
	0.1	$0.72955689 \times 10^{-3}$
	0.2	$0.82830031 \times 10^{-3}$
	0.3	$0.93649294 \times 10^{-3}$
	0.4	$0.10541347 \times 10^{-2}$

It can be seen from table (4.3) that as α increases $X(\alpha,\delta)$ decreases for a fixed δ whereas it increases as δ increases when α is held fixed.

4.5 DISPERSION OF A SOLUTE IN A FLUID FLOWING IN A PIPE WITH COMBINED HOMOGENEOUS AND HETEROGENEOUS REACTION

In this section we study the dispersion process by taking into account the effects of chemical reaction in the fluid and at the wall of the tube.

The diffusion equation in this case remains the same as before, i.e.,

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial c}{\partial \eta}\right) - \alpha^2 c = H\,[G_o - f(\eta)] \tag{4 64}$$

The boundary conditions become

$$\frac{\partial c}{\partial \eta} = 0 \qquad \eta = 0 \tag{4 65}$$

$$\frac{\partial c}{\partial \eta} + \beta c = 0 \text{ at } \eta = 1 \tag{4 66}$$

where $\beta = sR$ and s is the reaction rate constant

Solving equation (4 64) and using the boundary conditions, (4 65) and (4 66), we get

$$c = A\, I_o(\alpha\eta) - \frac{H\ G_o}{\alpha^2} - H\ \ F(\eta) \tag{4 67}$$

where H and $F(\eta)$ are given by equations (4 50-b) and (4 54) respectively and A is given by

$$A = \frac{H[\beta \frac{G_o}{\alpha^2} + \ \beta F(1) + \alpha S]}{\alpha I_1(\alpha) + \ \beta I_o(\alpha)} \tag{4 68}$$

$$F(1) = \int_0^1 \ [I_o(\alpha)\, K_o(\alpha t) - I_o(\alpha t)\, K_o(\alpha)]\ t\ f(t)\ dt \tag{4 69}$$

and S is given by equation (4.53)

The flux of the solute across a section of the tube is given by

$$Q = \frac{-\pi R^8}{2DL}\left(\frac{dp}{dz}\right)^2 \left(\frac{\partial c}{\partial \xi}\right)\ \ Y(\alpha,\beta) \tag{4 70}$$

where

$$Y(\alpha,\beta) = Y_2(\alpha) - Y_1(\alpha,\beta) \tag{4 71}$$

$$Y_1(\alpha,\beta)=\left[\frac{\beta\frac{G_o}{2}+\beta F(1)+\alpha S}{\alpha I_1(\alpha)+\beta I_o(\alpha)}\right]\int_o^1 \eta\,[G_o - f(\eta)]\, I_o(\alpha\eta)\,d\eta \qquad (4\ 72)$$

and

$$Y_2(\alpha) = \int_o^1 F(\eta)\,[G_o - f(\eta)]\,\eta \quad d\eta \qquad (4\ 73)$$

Comparing with Fick's law of diffusion, the effective dispersion coefficient is given by

$$D^* = \frac{R^6}{2D}\left(\frac{dp}{dz}\right)^2 \quad Y(\alpha,\beta) \qquad (4\ 74)$$

In order to study specifically the effect of viscosity variation on dispersion in the presence of homogeneous and heterogeneous reactions, we assume that the viscosity variation is given by equation (4 63) Polynomial expansions of the Bessel functions have been used to evaluate the integrals in $Y(\alpha,\beta,\delta)$ by neglecting α^4 and higher powers The values of $Y(\alpha,\beta,\delta)$ have then been calculated and tabulated in table (4 4)

<u>Table 4 4</u> ($\alpha = 0\ 2$)

β	δ	$\bar{I}(\alpha,\beta,\delta) = \mu_0^2\, Y(\alpha,\beta,\delta)$
	0 00	$0\ 64717996 \times 10^{-3}$
	0 01	$0\ 65577342 \times 10^{-3}$
	0 1	$0\ 73738613 \times 10^{-3}$
2 0	0 2	$0\ 83708411 \times 10^{-3}$
	0 3	$0\ 94627387 \times 10^{-3}$
	0 4	$0\ 10649554 \times 10^{-2}$
	0 00	$0\ 64716943 \times 10^{-3}$
	0 01	$0\ 65576275 \times 10^{-3}$
	0 1	$0\ 73737410 \times 10^{-3}$
4 0	0 2	$0\ 83707049 \times 10^{-3}$
	0 3	$0\ 94625855 \times 10^{-3}$
	0 4	$0\ 10649382 \times 10^{-2}$
	0 00	$0\ 64716586 \times 10^{-3}$
	0 01	$0\ 65575913 \times 10^{-3}$
	0 1	$0\ 73737005 \times 10^{-3}$
6 0	0 2	$0\ 83706590 \times 10^{-3}$
	0 3	$0\ 94625341 \times 10^{-3}$
	0 4	$0\ 10649325 \times 10^{-2}$

β	δ	$\bar{Y}(\alpha,\beta,\delta)$
8 0	0 00	$0\ 64716411 \times 10^{-3}$
	0 01	$0\ 65575736 \times 10^{-3}$
	0 1	$0\ 73736805 \times 10^{-3}$
	0 2	$0\ 83706362 \times 10^{-3}$
	0 3	$0\ 94625084 \times 10^{-3}$
	0 4	$0\ 10649296 \times 10^{-2}$
10 0	0 00	$0\ 64716306 \times 10^{-3}$
	0 01	$0\ 65575627 \times 10^{-3}$
	0 1	$0\ 73736683 \times 10^{-3}$
	0 2	$0\ 83706222 \times 10^{-3}$
	0 3	$0\ 94624929 \times 10^{-3}$
	0 4	$0\ 10649279 \times 10^{-2}$

It can be seen from table (4 4) that the effective dispersion coefficient decreases as β increases whereas it increases with increasing δ for a fixed β Similar result has been obtained in the case of parallel plates.

4 6 CONCLUSIONS

The effect of viscosity variation on Taylor diffusion has been investigated in the cases of flow between parallel plates and circular pipes by taking into consideration the homogeneous and heterogeneous reactions of the solute with the fluid. It is found in the above cases that the decrease of viscosity towards the wall increases the effective dispersion coefficient

It is also observed that for a given viscosity distribution, the effective dispersion coefficient decreases with the increasing reaction rate constants within the fluid and at the wall

NOMENCLATURE

c	Concentration of the solute in the fluid
D	olecular diffusion coefficient
D^*	Taylor's equivalent dispersion coefficient
h	Half of the width o. the channel
K	Reaction rate constant in the fluid
Q	Flux of the solute across any section of the channel or tube
R	Radius of the circular tube
u	Velocity of the fluid between parallel plates
$\bar{u}$	Average velocity of the fluid in the parallel plate case
u_x	Relative velocity of the fluid in parallel plate case with respect to a plane moving with the mean speed of the flow
W	Velocity of the fluid in circular tube
$\bar{W}$	Average velocity of the fluid in the circular tube
W_x	Velocity of the fluid in circular case relative to a plane moving with the mean speed of the flow
α	Non-dimensional reaction rate constant in the fluid
β	Non-dimensionalised reaction rate constant at the wall
μ	Viscosity of the fluid
μ_o	Viscosity of the fluid at the central line
ξ	Non-dimensionalised axial distance
η	Non-dimensionalised transverse distance

REFERENCES

1 Crank, J (1975)
The mathematics of diffusion (second edition)
Clarendon Press Oxford

2 Crank, J and Park, G S (eds) (1968)
Diffusion in Polymers
Academic Press

3 Crank, J and Gupta, R S (1972)
A moving boundary problem arising from the diffusion of oxygen in absorbing tissue
J Inst Math and its appl Vol 10, p 19

4 Cumming, G , Crank, J and Parker, I (1966)
Gaseous diffusion in airways of human lung
Resp Physiol , Vol 1, p 58

5 Fan, L.T and Hwang, W S (1965)
Dispersion of Ostwald de waele fluid in laminar flow through a cylinderical tube
Proc Roy Soc , Series A, Vol 283, p 576

6 Fan, L.T and Wang, C B (1966)
Dispersion of matter in non-Newtonian laminar flow through circular tubes
Proc Roy Soc , Series A, Vol 292, p 203

7. Ghoshal, S (1971)
Dispersion of solutes in non-Newtonian flows through a circular tube
Chem Engg Sc , Vol 26, p 185

8 Gill, W N and Sankarasubramanian, R (1970)
Exact analysis of unsteady convective diffusion.
Proc Roy Soc , Series A, Vol 316, p 341

9. ________________ (1971)
Dispersion of non-uniform slug in time dependent flow
Proc Roy Soc , Series A, Vol 322, p 101

10 Gupta, P S and Gupta, A S (1972)
Effect of homogeneous and heterogeneous reactions on the dispersion of a solute in the laminar flow between two plates
Proc Roy Soc , Series A, Vol 330, p 59

11 Haynes, R H (1960)
Physical basis of the dependence of blood viscosity on tube radius
Am J Physiology, Vol 198, p 1193

12 Haynes, R.H and Burton, A C (1959)
Role of non-Newtonian behaviour in hemodynamics
Am J Physiology, Vol 197, p 943

13 Lih, M M. (1969)
A mathematical model for the axial migration of suspended particles in tube flow
Bull Math Biophysics, Vol 31, p 143

14 Lih, M M (1975)
Transport phenomenon in medicine and biology
John Wiley and Sons, New York

15 Middleman, S (1972)
Transport phenomena in the cardiovascular system
Wiley Interscience, New York

16 Shah, S N and Cox, K E (1974)
Dispersion of solutes in non-Newtonian laminar flow through a circular tube Eyring model fluid.
Chem Engg Sc , Vol 29, p 1282

17 Taylor, Sir Geoffrey (1953)
Dispersion of soluble matter in solvent flowing slowly through a tube
Proc Roy Soc , Series A, Vol 219, p 186

18 ______________ (1954)
Dispersion of matter in turbulent flow through a pipe
Proc Roy Soc , Series A, Vol 223, p 446

19 ______________ (1954)
Conditions under which dispersion of a solute in a stream of solvent can be used to measure molecular diffusion.
Proc Roy. Soc , Series A, Vol 225, p 473

20 Whitmore, R L. (1968)
Rheology of Circulation
Pergamon Press

21 Wooding, R A. (1960)
Instability of viscous liquid of variable density in a vertical Hele-Shaw Cell
J Fluid Mech , Vol 7, p. 501

CHAPTER V

DISPERSION IN CASSON FLUID

5 1 INTRODUCTION

In Chapter IV, we have studied the dispersion through a physiological Newtonian fluid with variable viscosity, flowing through a rectangular channel and a circular pipe In general, biological fluids such as blood, do not behave as Newtonian fluids [Cokelet et al (1963), Merrill et al (1963), Charm and Kurland (1962 , 1965)] and can be represented by a non-Newtonian Casson model fluid at low shear rates. (See Chapter I)

As pointed out in Chapter IV the dispersion through non-Newtonian fluids flowing through ducts have have been studied by different investigators by considering different fluid models, power-law fluid, Fan and Hwang (1965), Bingham and Ellis models, Fan and Wang (1966), Reiner philippoff model, Ghoshal (1971), Eyring model, Shah and Cox (1974)

Keeping this in view, in this Chapter we study the dispersion of soluble matter in a Casson model fluid, following the approach of Taylor (1953)

5 2 DISPERSION IN CASSON MODEL FLUID FLOWING BETWEEN TWO PLATES

Consider the laminar flow of casson model fluid under a constant pressure gradient between two parallel plates distant 2h apart. The

physical configuration and the coordinate system are shown in fig. (5.1)

The equation of motion for one dimensional steady laminar flow of Casson model fluid is

$$-\frac{dp}{dx} - \frac{d\tau}{dy} = 0 \tag{5.1}$$

The stress-strain law is given by

$$\tau^{1/2} = \mu\left(-\frac{dV}{dy}\right)^{1/2} + \tau_o^{1/2} \quad (\tau \geq \tau_o)$$

$$\frac{dV}{dy} = 0, \; \tau \leq \tau_o \tag{5.2}$$

where τ_o is the yield stress of the fluid. Integrating equation (5.1) and using the boundary condition

$$\tau = 0 \qquad \text{at} \quad y = 0$$

we get

$$\tau = \left(-\frac{dp}{dx}\right) y \tag{5.3}$$

If y_o is the height of the core then we have

$$\tau_o = \left(-\frac{dp}{dx}\right) y_o \tag{5.4}$$

Eliminating τ from equations (5.2) and (5.3) and integrating with no slip condition at the wall, we get

$$V = \frac{1}{\mu^2}\left[\frac{P}{2}(h^2 - y^2) + \tau_o(h - y) - \frac{4}{3}\sqrt{P\tau_o}\,(h^{3/2} - y^{3/2})\right] \tag{5.5}$$

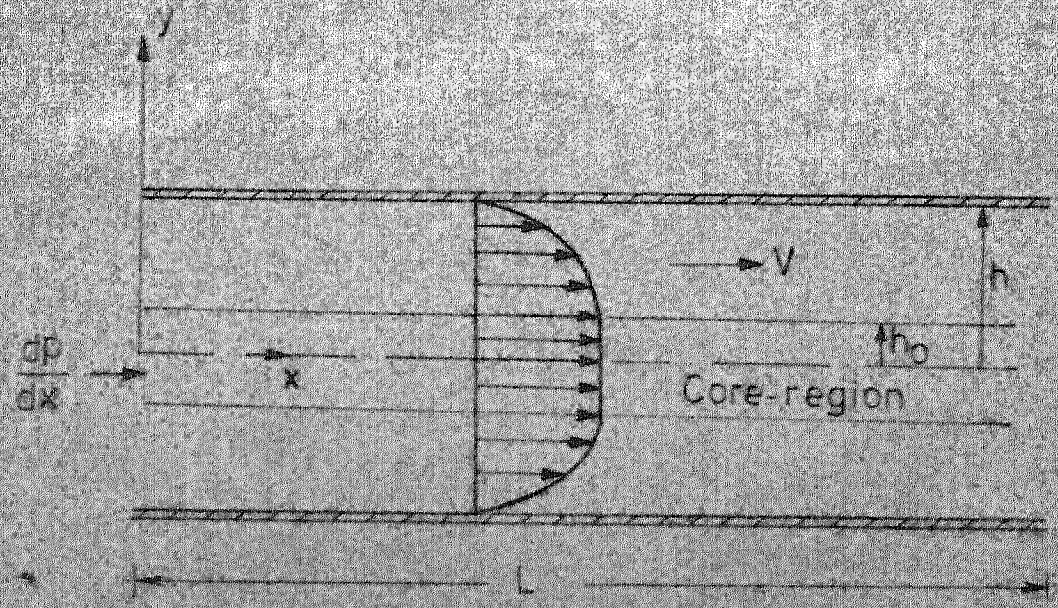

FIG. 5.1 PARALLEL PLATE GEOMETRY

in the region $(y_o \leq y \leq h)$ where $P = -\frac{dp}{dx}$ The velocity of the core is given by

$$V_c = \frac{1}{\mu^2}\left[\frac{P}{2}(h^2-y_o^2) + \tau_o(h-y_o) - \frac{4}{3}\sqrt{P\tau_o}\,(h^{3/2} - y_o^{3/2})\right] \quad (5\ 6)$$

in the region $(0 \leq y \leq y_o)$

Substituting

$$\eta = y/h\ , \quad \eta_o = y_o/h \quad (5\ 7)$$

in equations (5 5) and (5 6) and dividing we obtain

$$\frac{V}{V_c} = \frac{(1-\eta^2) + 2\eta_o\,(1-\eta) - \frac{8}{3}\eta_o^{1/2}\,(1-\eta^{3/2})}{g_o} \quad (\eta_o \leq \eta \leq 1) \quad (5\ 8)$$

where

$$g_o = 1 + 2\eta_o - \frac{8}{3}\eta_o^{1/2} - \frac{1}{3}\eta_o^2 \quad (5\ 9)$$

and

$$\frac{V}{V_c} = 1 \qquad (0 \leq \eta \leq \eta_o) \quad (5.10)$$

The average velocity $\bar{V}$, which is defined by

$$\bar{V} = \frac{1}{2h}\int_{-h}^{h} V\ dy \quad (5\ 11)$$

is obtained as

$$\bar{V} = m\,V_c \quad (5\ 12)$$

where

$$m = \frac{1}{g_o}\left(\frac{2}{3} + \eta_o - \frac{24}{15}\eta_o^{1/2} - \frac{1}{15}\eta_o^3\right) \quad (5.13)$$

The equation governing the concentration c of the solute is given by

$$\frac{\partial c}{\partial t} + V \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial y^2} \qquad (5.14)$$

Relative to a plane moving with the mean speed of the flow, the diffusion equation becomes

$$\frac{\partial c}{\partial t} + (V - \bar{V}) \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial y^2} \qquad (5.15)$$

Nondimensionalising by

$$\eta = \frac{y}{h}, \quad \xi = \frac{x - \bar{V} t}{L}, \quad \theta = \frac{t}{\bar{t}}, \quad \bar{t} = \frac{L}{\bar{V}}$$

and following Taylor's approach the diffusion equation becomes

$$\frac{\partial^2 c}{\partial \eta^2} = \frac{h^2}{DL} \left(\frac{\partial c}{\partial \xi}\right) V_x \qquad (5.16)$$

where $V_x = V - \bar{V}$, and $\frac{\partial c}{\partial \xi}$ is assumed constant.

From equations (5.8), (5.10) and (5.12), V_x can be obtained as follows

$$V_x = \bar{V} \left[\left(\frac{1}{m} - 1\right) - \frac{(\eta^2 - \eta_o^2) + 2\eta_o(\eta - \eta_o) - \frac{8}{3}\eta_o^{1/2}(\eta^{3/2} - \eta_o^{3/2})}{m\, g_o}\right] \qquad (5.17)$$

in region $(\eta_o \leq \eta \leq 1)$

$$V_x = \bar{V}\left(\frac{1}{m} - 1\right) \qquad (5.18)$$

in region $(0 \leq \eta \leq \eta_o)$

where g_o and m are given by equations (5.9) and (5.13)

The boundary conditions are

(i) $\frac{\partial c}{\partial \eta} = 0$ at $\eta = 0$ (5 19a)

(ii) $\frac{\partial c}{\partial \eta} = 0$ at $\eta = 1$ (5 19b)

(iii) $c = c_o$ at $\eta = 0$ (5 19c)

(iv) c is continuous at $\eta = \eta_o$ (5 19d)

Solving equation (5 16) using equations (5 17) through (5 19d), we get the concentration c of the solute as follows

$$c = c_o + \frac{h^2}{DL} \frac{\partial c}{\partial \xi} \bar{V} \left[\left(\frac{1}{m} - 1 \right) \frac{\eta^2}{2} - \frac{1}{m g_o} f(\eta) \right] \qquad (5\ 20)$$

$$(\eta_o \leq \eta \leq 1)$$

$$c = \frac{h^2}{DL} \frac{\partial c}{\partial \xi} \bar{V} \left[\left(\frac{1}{m} - 1 \right) \frac{\eta^2}{2} \right] + c_o \qquad (5\ 21)$$

$$(0 \leq \eta \leq \eta_o)$$

where

$$f(\eta) = \frac{\eta^4}{12} - \frac{\eta_o \eta^3}{3} - \frac{32}{105} \eta_o^{1/2} \eta^{7/2} - \frac{1}{6} \eta_o^2 \eta^2 + \frac{1}{15} \eta_o^3 \eta - \frac{1}{84} \eta_o^4 \qquad (5\ 22)$$

The volumetric flow rate at which the solute is transported across a section of the channel of unit breadth is

$$Q = 2 \int_0^{\eta_o} c V_x \, d\eta + 2 \int_{\eta_o}^{1} c V_x \, d\eta \qquad (5\ 23)$$

Substituting appropriate expressions for c and V_x and integrating, we obtain

$$C = \frac{2h^3}{\mu L}\left(\frac{\partial c}{\partial \xi}\right)\frac{\bar{V}^2}{m^2}\left[\frac{(1-m)^2\eta_o^3}{6} + \frac{I}{g_o^2}\right] \qquad (5\ 24)$$

where

$$I = \frac{E^2}{6} - \frac{7E}{60} - \frac{E\eta_o}{3} + \frac{344\ E\ \eta_o^{1/2}}{945} - \frac{E\ \eta_o^3}{30}$$

$$+ \frac{E\eta_o^4}{84} + \frac{1}{84} + \frac{\eta_o}{12} - \frac{332}{4095}\eta_o^{1/2} + \frac{\eta_o^3}{60}$$

$$+ \frac{153}{3780}\eta_o^4 + \frac{2\eta_o^2}{15} - \frac{944}{3465}\eta_o^{3/2} - \frac{\eta_o^5}{84}$$

$$+ \frac{128}{945}\eta_o - \frac{16}{315}\eta_o^{7/2} + \frac{4}{315}\eta_o^{9/2} - \frac{E^2\eta_o^3}{6}$$

$$+ \frac{29E\ \eta_o^5}{270} - \frac{1363\ \eta_o^7}{77220} \qquad (5\ 25)$$

and

$$E = \frac{1}{3} + \eta_o - \frac{16}{15}\eta_o^{1/2} + \frac{1}{15}\eta_o^3 \qquad (5\ 26)$$

Comparing with Fick's law of diffusion, the effective dispersion coefficient is given by

$$D^* = -\frac{h^2\ \bar{V}^2}{D\ m^2} F(\eta_o) \qquad (5\ 27)$$

where

$$F(\eta_o) = \frac{(1-m)^2\eta_o^3}{6} + \frac{I}{g_o^2}$$

When $\eta_o = 0$, the effective dispersion coefficient is given by

$$D^*_{\eta=0} = \frac{h^6(dp/dx)^2}{\mu^2\ D}\left(\frac{2}{945}\right) \qquad (5\ 28)$$

which agrees with the result of Wooding (1960) If $\eta_o << 1$, then η_o and higher powers of η_o are small when compared to $\eta_o^{1/2}$. With this approximation, D^* is approximated as

$$D^* = \frac{a^2 \bar{V}^2}{D} \; \frac{2}{105} \left(1 - \frac{128}{195} \eta_o^{1/2}\right) \qquad (5\ 29)$$

It may be concluded from equation (5 29) that the non-Newtonian character of the fluid decreases the effective dispersion coefficient

To study the effect of non-Newtonian character on effective dispersion coefficient, $\frac{1}{m^2}\,[F(\eta_o)]$ is calculated and tabulated in table (5 1). It can be seen from table (5 1) that as η_o increases the effective dispersion coefficient decreases

5 3 DISPERSION IN CASSON MODEL FLUID IN THE PRESENCE OF HOMOGENEOUS REACTION

In this case we assume that the solut , while diffusing, undergoes an irreversible chemical reaction in the fluid. The diffusion equation then becomes

$$\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial y^2} - Kc \qquad (5\ 30)$$

Following Taylor's approach, as in the last section, the differential equation governing the distribution of the soluble matter relative to a plane moving with the mean speed of the fluid is obtained as

$$\frac{\partial^2 c}{\partial \eta^2} - \alpha^2 c = \frac{h^2}{DL}\left(\frac{\partial c}{\partial \xi}\right) V_x \qquad (5\ 31)$$

where

$$\alpha^2 = \frac{Kh^2}{D} \qquad (5\ 32)$$

and V_x is given by equations (5 17) and (5 18) in the two regions

The boundary conditions are

(i) $\frac{\partial c}{\partial \eta} = 0$ at $\eta = 0$ (5 33a)

(ii) $\frac{\partial c}{\partial \eta} = 0$ $\eta = 1$ (5 33b)

(iii) c and $\frac{\partial c}{\partial \eta}$ are continuous at $\eta = \eta_o$ (5 33c)

Solving the equation (5 31) with the boundary condition (5 33a) to (5 33c) we obtain the concentration of the solute as follows

$$c = H\,[A_2 \cosh \alpha\eta + B_2 \sinh \alpha\eta + F(\eta)]$$
$$(\eta_o \leq \eta \leq 1) \qquad (5\ 34)$$

where,

$$A_2 = \frac{1}{\sinh \alpha}\,(M_o - P_o \cosh \alpha\eta_o \cosh \alpha + N_o \sinh \alpha\eta_o \cosh \alpha)$$
$$(5\ 35)$$

$$B_2 = P_o \cosh \alpha\eta_o - N_o \sinh \alpha\eta_o \qquad (5\ 36)$$

$$H = \frac{h^2}{DL}\,\bar{V}\left(\frac{\partial c}{\partial \xi}\right) \qquad (5\ 37)$$

$$\alpha M_o = \frac{-8\eta_o^{1/2}}{3m\,g_o} \int_{\eta_o}^{1} x^{3/2} \cosh(\alpha - \alpha x)\,dx - \frac{2(1+\eta_o)}{m\,\alpha^2\,g_o} \qquad (5\ 38)$$

$$I_o = -\left[\frac{8\eta_o^2}{3\alpha^2 m\, g_o} + \frac{2}{m\,\alpha^4 g_o}\right] \tag{5 39}$$

$$P_o = -\frac{4\eta_o}{m\,\alpha^3 g_o} \tag{5 40}$$

$$F(\eta) = \frac{1}{3m\alpha^4 g_o}\left[3\alpha^2(\eta+\eta_o)^2 + 6 - 4\alpha^2\eta_o^2\right]$$

$$-\frac{1}{\alpha^2}\left(\frac{1}{m}-1\right) + \frac{8\eta_o^{1/2}}{3m\alpha g_o}\int_{\eta_o}^{\eta} x^{3/2}\sinh(\alpha\eta - \alpha x)\,dx \tag{5 41}$$

and

$$c = A_1 H \cosh\alpha\eta - \frac{H}{\alpha^2}\left(\frac{1}{m}-1\right) \quad (0 \leq \eta \leq \eta_o) \tag{5 42}$$

where

$$A_1 = I_o \operatorname{cosech}\alpha - P_o \operatorname{cosech}\alpha\eta_o$$
$$-(P_o \cosh\alpha\eta_o - N_o \sinh\alpha\eta_o)(\coth\alpha - \coth\alpha\eta_o) \tag{5 43}$$

and M_o, P_o, N_o are same as in equations (5 38) to (5 40)

When $\eta_o = 0$, we get the concentration c as

$$c = \frac{h^4}{DL^2}\left(\frac{\partial c}{\partial \xi}\right)\left(\frac{\partial p}{\partial \xi}\right)\frac{\cosh\alpha\eta}{\mu\alpha^3 \sinh\alpha} + \frac{h^4}{2\mu DL^2}\left(\frac{\partial c}{\partial \xi}\right)\left(\frac{\partial p}{\partial \xi}\right)\frac{1}{\alpha^2}\left[\frac{1}{3} - \frac{2}{\alpha^2} - \eta^2\right] \tag{5 44}$$

which agrees with the result of Gupta et al. (1972)

The volumetric flow rate Q at which the solute is transported across a section of the channel of unit breadth is given by the equation (5 23)

Substituting the appropriate expressions for c and V_x from equations (5 34) and (5 42), (5 17) and (5 18) we obtain the flux Q as follows

$$Q = -\frac{2h^3 V^2}{DL}\left(\frac{\partial c}{\partial \xi}\right) F(\alpha,\eta_o) \tag{5 44}$$

where

$$F(\alpha,\eta_o) = \frac{(1-m)^2\eta_o}{m^2\alpha^2} - \frac{A_1(1-m)\sinh\alpha\eta_o}{m\,\alpha} - \frac{J}{m^2\alpha^2 g_o^2} \tag{5 46}$$

and J is given by the following equation, m and A_1 are same as in equations (5.13) and (5 43).

$$J = (\sinh\alpha)\left\{\frac{\phi_1\varepsilon_1}{\alpha} - \frac{(1+\eta_o)^2\varepsilon_1}{\alpha} - \frac{2\varepsilon_1}{\alpha^3} - \frac{2(1+\eta_o)\varepsilon_2}{\alpha^2}\right\}$$

$$+(\cosh\alpha)\left\{\frac{\phi_1\varepsilon_2}{\alpha} + \frac{2(1+\eta_o)\varepsilon_1}{\alpha^2} - \frac{(1+\eta_o)^2\varepsilon_2}{\alpha} - \frac{2\varepsilon_2}{\alpha^3}\right\}$$

$$+\frac{8}{3}\eta_o^{1/2}\int_{\eta_o}^{1}(\varepsilon_1\cosh\alpha\eta + \varepsilon_2\sinh\alpha\eta)\eta^{3/2}\,d\eta$$

$$+\frac{8}{3}\eta_o^{1/2}\alpha^2\int_{\eta_o}^{1}\left[\phi_1-(\eta+\eta_o)^2+\frac{8}{3}\eta_o^{1/2}\eta^{3/2}\right]T(\eta)\,d\eta$$

$$-(\sinh\alpha\eta_o)\left(\frac{\phi_1\varepsilon_1}{\alpha} - \frac{4\eta_o^2\varepsilon_1}{\alpha} - \frac{2\varepsilon_1}{\alpha^3} + \frac{4\eta_o\varepsilon_2}{\alpha^2}\right)$$

$$-(\cosh\alpha\eta_o)\left(\frac{\phi_1\varepsilon_2}{\alpha} + \frac{4\eta_o\varepsilon_1}{\alpha^2} - \frac{4\eta_o^2\varepsilon_2}{\alpha} - \frac{2\varepsilon_2}{\alpha^3}\right)$$

$$+\frac{8}{3}\eta_o^{1/2}\left[\frac{2}{9}+\frac{2}{5}\varepsilon_3+\frac{4\eta_o}{7}+\frac{2}{5}\eta_o^2\right]$$

$$+\left(\phi_1\varepsilon_3+\frac{\phi_1}{3}-\frac{\varepsilon_3}{3}-\frac{1}{5}\right)+\eta_o\left(\phi_1-1-\varepsilon_3-\phi_1\varepsilon_3\right)$$

$$-\eta_o^2\left(2-\phi_1+\varepsilon_3\right)+\eta_o^3\left(3\phi_1-2+\frac{19}{15}\varepsilon_3\right)$$

$$-\eta_o^4-\frac{1849}{189}\eta_o^5 \tag{5.47}$$

and

$$\varepsilon_1 = m\alpha^2 g_o A_2 \tag{5 48a}$$

$$\varepsilon_2 = m\alpha^2 g_o B_2 \tag{5 48b}$$

$$\varepsilon_3 = \frac{2}{\alpha^2} - \frac{4\eta_o^2}{3} - (1-m)\,g_o \tag{5 48c}$$

$$\phi_1 = (1-m)\,g_o + \frac{4\eta_o^2}{3} \tag{5 48d}$$

$$T(\eta) = \int_{\eta_o}^{\eta}\frac{\sinh\alpha(\eta-x)}{\alpha}x^{3/2}\,dx \tag{5 48e}$$

Comparing with Fick's law of diffusion the e uivalent dispersion coefficient is given by

$$D^* = \frac{h^2\bar{V}^2}{D} F(\alpha, \eta_o) \qquad (5\ 49)$$

where $F(\alpha, \eta_o)$ is given by equation (5 46)

When $\eta_o = 0$, from equation (5 49), D^* can be obtained as follows

$$D^*_{\eta=0} = \frac{9h^2\bar{V}^2}{D} \frac{1}{\alpha^2} \left[\frac{1}{45} - \frac{1}{3\alpha^2} + \frac{\coth\alpha}{\alpha^3} - \frac{1}{\alpha^4} \right] \qquad (5\ 50)$$

which agrees with the result of Gupta et al (1972)

When η_o is small such that η_o and higher powers are neglected in comparison to $\eta_o^{1/2}$, the equivalent dispersion coefficient reduces to

$$D_1^* = \frac{h^2\bar{V}^2}{D} F_1(\alpha, \eta_o) \qquad (5\ 51)$$

where

$$F_1(\alpha,\eta_o) = \frac{1}{m^2\alpha^2 g_o^2} \Big[\Big(\frac{4}{45} - \frac{4}{3\alpha^2} + \frac{4\coth\alpha}{\alpha^3} - \frac{4}{45}\Big)$$

$$- \frac{8\eta_o^{1/2}}{3} \Big(\frac{2S}{3} + \frac{2S}{\alpha^2} - \frac{2S\coth\alpha}{\alpha} - \frac{2T\,\mathrm{cosech}\,\alpha}{\alpha}$$

$$+ \frac{4}{45} + \frac{4}{5\alpha^2} + \alpha^2 G^*\Big)\Big] \qquad (5\ 52)$$

and

$$S = \int_{\eta_o}^{1} t^{3/2} \cosh(\alpha - \alpha t)\, dt \qquad (5\ 53)$$

$$T = \int_{n_o}^{1} t^{3/2} \cosh(\alpha t)\, dt \qquad (5.54)$$

$$G^* = \int_{n_o}^{1} (\tfrac{1}{3} - t^2) \left[\int_{n_o}^{t} \frac{\sinh(\alpha t - \alpha s)}{\alpha} s^{3/2}\, ds \right] dt \qquad (5.55)$$

As $\alpha \to 0$, it can be shown that

$$\underset{\alpha \to 0}{Lt}\; D_1^* = \frac{h^2 \bar{V}^2}{D} \frac{2}{105} (1 - \frac{128}{195} n_o^{1/2}) \qquad (5.56)$$

which is same as equation (5.29)

To study the effects of homogeneous reaction on dispersion, equation (5.51) is calculated and tabulated in table (5.2). It can be seen from table (5.2) that as α increases, the equivalent dispersion coefficient decreases and this decrease is enhanced further by increasing n_o

5.4 DISPERSION IN CASSON FLUID FLOWING THROUGH A CIRCULAR TUBE

In the last two sections we have studied the dispersion of a solute in a casson model fluid flowing between two parallel plates. In the following we get corresponding results when the fluid is flowing through a circular pipe. The fully developed velocity profile in this case is given by, (see Fig. 5.2))

$$V = \frac{1}{\mu^2} \left[\frac{P}{4} (R^2 - r^2) + \tau_o (R - r) - \frac{4}{3} \sqrt{\frac{P \tau_o}{2}} (R^{3/2} - r^{3/2}) \right] \qquad (5.57)$$

in the region $(r_o \leq r \leq R)$ where $P = -\frac{dp}{dx}$ and r_o is the radius of the core, Whitmore (1968)

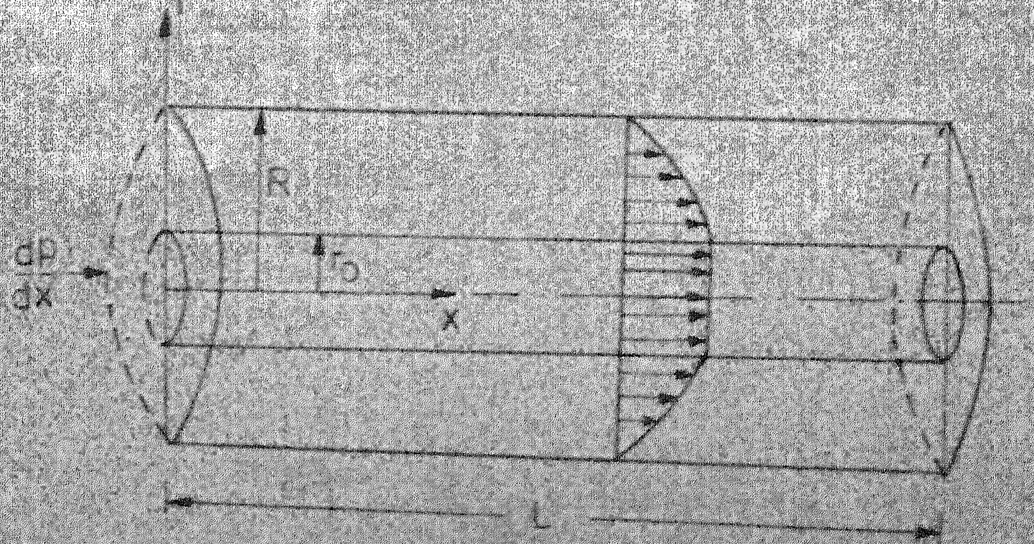

FIG.5.2 CYLINDRICAL TUBE GEOMETRY

The velocity of the core in the region $(0 \leq r \leq r_o)$ is given by,

$$W_c = \frac{1}{\mu^2}\left[\frac{P}{4}(R^2 - r_o^2) + \tau_o(R - r_o) - \frac{4}{3}\sqrt{\frac{P\tau_o}{2}}\,(R^{3/2} - r_o^{3/2})\right] \tag{5 58}$$

In this case, the yield stress τ_o and the radius of the core r_o are related by the relation, Bird et al (1960)

$$\tau_o = \left(-\frac{dp}{dx}\right)\frac{r_o}{2} \tag{5 59}$$

Substituting,

$\eta = r/R$, $\eta_o = r_o/R$ and dividing the equation (5 57) by equation (5 58), we get

$$\frac{W}{W_c} = \frac{(1-\eta^2) + 2\eta_o(1-\eta) - \frac{8}{3}\eta_o^{1/2}(1-\eta^{3/2})}{g_o} \tag{5 60}$$

in the region $(\eta_o \leq \eta \leq 1)$, and

$$\frac{W}{W_c} = 1 \tag{5 61}$$

in the region $(0 \leq \eta \leq \eta_o)$, where g_o is given by equation (5 9)

The average velocity, which is defined by,

$$\bar{W} = 2\int_0^1 \eta W \, d\eta \tag{5 62}$$

is obtained as

$$\bar{W} = n\,W_c \tag{5 63}$$

where

$$n = \frac{1}{g_o}\left[\frac{1}{2} + \frac{2\eta_o}{3} - \frac{8}{7}\eta_o^{1/2} - \frac{1}{42}\eta_o^4\right] \tag{5 64}$$

Following Taylor's approach (1953), the governing equation for concentration of the solute relative to a plane moving with the mean speed of the fluid is given by

$$\frac{\partial^2 c}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial c}{\partial \eta} = \frac{R^2}{DL}\frac{\partial c}{\partial \xi} W_x \qquad (5\ 65)$$

where $W_x (= W - \bar{W})$ is the velocity of the fluid relative to the moving plane and is given by equations (5 60), (5 61) and (5 63) as follows

$$W_x = \bar{W}\left[\left(\frac{1}{\eta} - 1\right) - \frac{1}{\eta g_o}\left(\eta^2 + 2\eta_o \eta - \frac{8\eta_o^{1/2}}{3}\eta^{3/2} - \frac{1}{3}\eta_o^2\right)\right]$$

$$(\eta_o \leq \eta \leq 1) \qquad (5\ 66)$$

and

$$W_x = \bar{W}\left(\frac{1}{\eta} - 1\right) \quad (0 \leq \eta \leq \eta_o) \qquad (5\ 67)$$

The boundary conditions are

(i) $\frac{\partial c}{\partial \eta} = 0$ at $\eta = 0$ (5 68a)

(ii) $\frac{\partial c}{\partial \eta} = 0$ at $\eta = 1$ (5.68b)

(iii) $c = c_o$ at $\eta = 0$ (5 68c)

(iv) c is continuous at $\eta = \eta_o$ (5 68d)

Solving the equation (5.65) with the boundary conditions (5 68a) through (5 68d) we obtain the concentration of the solute as follows

$$c = c_o + H\left[\left(\frac{1-\eta}{\eta}\right)\frac{\eta^2}{4} - \frac{1}{\eta g_o} g(\eta)\right] \quad (\eta_o \leq \eta \leq 1) \qquad (5\ 69)$$

where

$$g(\eta) = \frac{\eta^4}{16} + \frac{2\eta_o \eta^3}{9} - \frac{32}{147} \eta_o^{1/2} \quad \eta^{7/2}$$

$$- \frac{\eta_o^2 \ \eta^2}{12} + \frac{\eta_o^4}{84} \ \log\ (\eta/\eta_o) + \frac{115\eta_o^4}{7056} \qquad (5\ 70)$$

$$\Psi = \frac{R^2}{DL} \ \frac{\partial c}{\partial \xi} \ \bar{W} \qquad (5\ 71)$$

and

$$c = \frac{R^2}{DL} \ \frac{\partial c}{\partial \xi} \bar{W} \quad [(\frac{1-n}{n}) \ \frac{\eta^2}{2}] + c_o \ (0 \leq \eta \leq \eta_o) \qquad (5\ 72)$$

The volumetric flow rate with which the solute is transported across a section of the tube is

$$Q = 2\pi R^2 \int_0^1 \eta \quad c \quad W_x \ d\eta \qquad (5\ 73)$$

Substituting the expressions for c and W_x from equations (5 69), (5 72), (5 66) and (5 67) we get

$$Q = \frac{2\pi R^2 \ H \ \bar{W}}{n^2} \quad [\frac{(1-n)^2}{16} \quad \eta_o^4 + \frac{A_o}{g_o^2}] \qquad (5\ 74)$$

where

$$A_o = 4\phi_o \ \phi_1(\frac{1}{2} - \frac{\eta_o^2}{2}) - 2\eta_o\phi_o \ (\frac{1}{3} - \frac{\eta_o^3}{3})$$

$$+ (4\phi_1^2 - \phi_o) \ (\frac{1}{4} - \frac{\eta_o^4}{4})$$

$$- \frac{26\eta_o}{9} (\phi_1) \ (\frac{1}{5} - \frac{\eta_o^5}{5})$$

$$+ (\frac{4\eta_o^2}{9} - \frac{5\phi_1}{4}) \ (\frac{1}{6} - \frac{\eta_o^6}{6})$$

$$+\frac{1091}{1176}\eta_o\left(\frac{1}{7}-\frac{\eta_o^7}{7}\right)+\left(\frac{1}{128}-\frac{\eta_o^8}{128}\right)$$

$$+\frac{8}{3}\phi_o\eta_o^{1/2}\left(\frac{2}{7}-\frac{2}{7}\eta_o^{7/2}\right)$$

$$+\frac{8}{3}\eta_o^{1/2}\left(\frac{65}{49}\phi_1\right)\left(\frac{2}{11}-\frac{2}{11}\eta^{11/2}\right)$$

$$-\frac{1360}{1323}\eta_o^{3/2}\left(\frac{2}{13}-\frac{2}{13}\eta^{13/2}\right)+\frac{115}{2205}\eta_o^{1/2}\left(1-\eta_o^{15/2}\right)$$

$$-\frac{\eta_o^4\phi_1}{84}\left(-1-\eta_o\log\eta_o+\eta_o\right)$$

$$+\frac{\eta_o^5}{21}\left(-\frac{1}{4}-\frac{\eta_o^2}{2}\log\eta_o+\frac{\eta_o^2}{4}\right)$$

$$-\frac{2\eta_o^{9/2}}{63}\left(-\frac{4}{25}-\frac{2}{5}\eta_o^{5/2}\log\eta_o+\frac{4\eta_o^{5/2}}{25}\right)$$

$$+\frac{\eta_o^4}{84}\left(-\frac{1}{9}-\frac{\eta_o^3}{3}\log\eta_o+\frac{\eta_o^3}{9}\right)\tag{5.75}$$

where

$$\phi_o=\frac{\eta_o^4}{84}\log\eta_o-\frac{115}{7056}\eta_o^4\tag{5 76}$$

$$\phi_1=\frac{1}{8}+\frac{\eta_o}{3}-\frac{8}{21}\eta_o^{1/2}+\frac{1}{168}\eta_o^4\tag{5 77}$$

Comparing with Fick's law of diffusion, the effective dispersion coefficient is obtained as

$$D^*=-\frac{R^2\bar{W}^2}{D}\left[\frac{(1-n)^2\eta_o^4}{8n^2}+\frac{2A_o}{n^2g_o^2}\right]=\frac{R^2\bar{W}^2}{D}F^*(\alpha,\eta_o)\tag{5 78}$$

When $\eta_o=0$, we get the effective dispersion coefficient as

$$D^* = \frac{R^2\bar{W}^2}{48D} \qquad (5\ 79)$$

which agrees with the result of Taylor (1953)

For $\eta_o << 1$ and neglecting η_o and its higher powers in comparison to $\eta_o^{1/2}$, D^* can be approximated as

$$D^* = \frac{R^2\bar{W}^2}{D}\left[\frac{1}{48} - \frac{38}{3465}\eta_o^{1/2}\right] \qquad (5\ 80)$$

Hence, it may be concluded in this case also that the non-Newtonian character of the flow decreases the **effective** dispersion coefficient

To study the effects of non-Newtonian behaviour on dispersion, equation (5.78) is calculated and tabulated in table (5.3). It can also be seen from the table (5 3) that as η_o increases, the **effective** dispersion coefficient decreases

5 5 DISPERSION IN CASSON MODEL FLUID FLOWING IN A CIRCULAR TUBE IN THE PRESENCE OF CHEMICAL REACTION

In this section we study the dispersion through a casson fluid by considering the effects of an irreversible chemical reaction in the fluid.

The differential equation governing the concentration is modified as follows:

$$\frac{\partial^2 c}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial c}{\partial \eta} - \alpha^2 c = H\left(\frac{1}{\eta} - 1\right) \quad (0 \leq \eta \leq \eta_o) \qquad (5\ 81)$$

$$\frac{\partial^2 c}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial c}{\partial \eta} - \alpha^2 c = H\left[\left(\frac{1}{\eta} - 1\right) - \frac{G(\eta)}{\eta g_o}\right] (\eta_o \leq \eta \leq 1) \qquad (5\ 82)$$

where

$$G(\eta) = \eta^2 + 2\eta_b \eta - \frac{8}{3}\eta_o^2 \ \eta^{3/2} - \frac{1}{3}\eta_o^2 \qquad (5\ 83)$$

The boundary conditions are

(i) $\frac{\partial c}{\partial \eta} = 0$ at $\eta = 0$ (5 84a)

(ii) $\frac{\partial c}{\partial \eta} = 0$ at $\eta = 1$ (5 84b)

(iii) c and $\frac{\partial c}{\partial \eta}$ are continuous

at $\eta = \eta_o$ (5 84c)

Solving the equation (5 81) and (5 82) and using the boundary conditions (5 84a) to (5 84c), we obtain the concentration as follows

$$c = \frac{H\ S_1\ I_o(\alpha\eta)}{n g_o I_1(\alpha)} - \frac{H\ (1-n)}{n\alpha^2} \quad (0 \leq \eta \leq \eta_o) \qquad (5\ 85)$$

$$c = \frac{H\ S_1 I_o(\alpha\eta)}{n\ g_o I_1(\alpha)} - \frac{H(1-n)}{n\alpha^2} - \frac{H}{n g_o} F_o(r) \quad (\eta_o \leq \eta \leq 1) \qquad (5\ 86)$$

where

$$S_1 = \int_{\eta_o}^{1} [I_1(\alpha)\, K_o(\alpha t) + I_o(\alpha t)\, K_1(\alpha)]\ tG(t)dt \qquad (5\ 87)$$

$$F_o(\eta) = \int_{\eta_o}^{\eta} [I_o(\alpha\eta)\, K_o(\alpha t) - I_o(\alpha t)\, K_o(\alpha\eta)]\, tG(t)\ dt \qquad (5\ 88)$$

and H and n are given by equations (5 71) and (5 64) respectively

The volumetric rate at which the solute is transported across any section of the tube is given by equation (5 73) Substituting appropriate expressions for c and W_x from equations(5 85), (5 86), (5 66) and (5.67) we obtain Q as

$$Q = \frac{-2\pi R^4 \bar{W}^2}{DL} \left(\frac{\partial c}{\partial \xi}\right) Y_o(\alpha, \eta_o) \tag{5 89}$$

where

$$Y_o(\alpha,\eta_o) = \frac{1}{n^2 g_o^2} \Big[\frac{S_1}{I_1(\alpha)} \int_{\eta_o}^{1} t\, G(t)\, I_o(\alpha t)\, dt - \frac{(1-n)\, g_o S_1}{\alpha} + (1-n)\, g_o \int_{\eta_o}^{1} t\, F_o(t)\, dt - \int_{\eta_o}^{1} t F_o(t)\, G(t)\, dt \Big] \tag{5 90}$$

Comparing with Fick's law of diffusion, the effective dispersion coefficient is obtained as

$$D^* = \frac{2R^2 \bar{W}^2}{D} Y_o(\alpha, \eta_o) \tag{5 91}$$

When $\eta_o = 0$, we obtain the effective dispersion coefficient from equation (5 91) as

$$D^*_{\eta=0} = \frac{R^2 \bar{W}^2}{D} \left[\frac{32\, I_2(\alpha)}{\alpha^5\, I_1(\alpha)} + \frac{1}{3\alpha^2} - \frac{8}{\alpha^4} \right] \tag{5 92}$$

which is same as obtained in Chapter IV, equation (4 60)

When $\alpha \to 0$, and η_o is small so that η_o and higher powers can be neglected in comparison to $\eta_o^{1/2}$, we obtain the effective

dispersion coefficient as

$$D^*_{\alpha=0} = \frac{R^2\bar{W}^2}{D} \left[\frac{1}{48} - \frac{38}{3465} \eta_o^{1/2}\right] \tag{5 93}$$

which is the same as in equation (5 80)

To study the effects of homogeneous reaction on dispersion, the equation (5 90) is calculated and tabulated in table (5 4) It can be seen from table (5 4) that as α increases, the effective dispersion coefficient decreases and this decrease is enhanced further by the increasing η_o

5 6 CONCLUSIONS

In this chapter, dispersion of a soluble matter through a casson model fluid flowing in a rectangular channel and circular pipe has been studied

It is found that the effective dispersion coefficient in the casson fluid is less than that of Newtonian fluid. When $\eta_o = 0$, the results are found to agree with those of Gupta et. al (1972), Wooding (1960) and Taylor (1953) It is also observed that in the presence of homogeneous reaction, the equivalent dispersion coefficient decreases when the reaction rate constant increases This decrease is enhanced further by the increasing non-Newtonian character of the fluid for a given mean velocity

Table 5 1

n_o	$\frac{1}{m^2}F(n_o)$
0 00	0 19047619 × 10^{-1}
0 02	0 17070296 × 10^{-1}
0 04	0 16116019 × 10^{-1}
0 06	0 15324750 × 10^{-1}
0 08	0 14616516 × 10^{-1}
0 1	0 13960928 × 10^{-1}
0 2	0 11110413 × 10^{-1}
0 3	0 86673169 × 10^{-2}
0 4	0 65135310 × 10^{-2}
0 5	0 46335787 × 10^{-2}
0 6	0.30405961 × 10^{-2}

Table 5.2

n_o	α	$F_1(\alpha,n_o)$
0.02	0.1	$0.17259826 \times 10^{-1}$
	0.2	$0.17201045 \times 10^{-1}$
	0.3	$0.17103077 \times 10^{-1}$
	0.4	$0.16965921 \times 10^{-1}$
0.04	0.1	$0.16527188 \times 10^{-1}$
	0.2	$0.16467729 \times 10^{-1}$
	0.3	$0.16368629 \times 10^{-1}$
	0.4	$0.16229890 \times 10^{-1}$
0.06	0.1	$0.15965015 \times 10^{-1}$
	0.2	$0.15905035 \times 10^{-1}$
	0.3	$0.15805068 \times 10^{-1}$
	0.4	$0.15665114 \times 10^{-1}$

Table 5 3

η_o	$F^*(\eta_o)$
0 02	$0\ 19082364 \times 10^{-1}$
0 04	$0\ 18221351 \times 10^{-1}$
0 06	$0\ 17496617 \times 10^{-1}$
0 08	$0\ 16837958 \times 10^{-1}$
0 1	$0\ 16218441 \times 10^{-1}$
0 2	$0\ 13377557 \times 10^{-1}$
0 3	$0\ 10668601 \times 10^{-1}$
0 4	$0.79830004 \times 10^{-2}$
0 5	$0.53748746 \times 10^{-2}$
0 6	$0\ 30072266 \times 10^{-2}$

Table 5 4

η_o	α	$Y_o(\alpha,\eta_o)$
0 01	0 1	0 49270213 $\times 10^{-2}$
	0 2	0 49055965 $\times 10^{-2}$
	0 3	0 48698883 $\times 10^{-2}$
	0 4	0 48198969 $\times 10^{-2}$
0 03	0 1	0 47236286 $\times 10^{-2}$
	0 2	0 46941452 $\times 10^{-2}$
	0 3	0 46450059 $\times 10^{-2}$
	0 4	0 45762112 $\times 10^{-2}$
0.05	0 1	0 45835926 $\times 10^{-2}$
	0 2	0 45485608 $\times 10^{-2}$
	0 3	0 44901745 $\times 10^{-2}$
	0 4	0 44084336 $\times 10^{-2}$

NOMENCLATURE

c	Concentration of the solute in the fluid
D	molecular diffusion coefficient
D^*	equivalent dispersion coefficient
h	height of the rectangular channel over the central line
K	reaction rate constant
L	characteristic length along the axis of the channel or tube
p	pre sure
Q	flux of the solute across a section of the channel or tube
R	radius of the tube
r_o	radius of the core
V	velocity in the non-core region of the rectangular channel
V_c	velocity of the core in the rectangular channel
V_x	velocity of the fluid relative to a plane moving with the mean speed of the fluid
$\bar{V}$	mean speed of the fluid in rectangular channel
W	velocity of the fluid in the non-core region of circular tube
W_c	velocity of the core in the flow through a circular tube
W_x	velocity of the fluid in the circular tube, relative to a plane moving with the mean speed of the flow
$\bar{W}$	mean speed of the flow in the circular tube
y_o	height of the core from the central line in the case of flow in a rectangular channel

μ	viscosity of the fluid
ξ	non-dimensionalised axial distance relative to a plane moving with the mean speed of the flow
η	non-dimensionalised transverse distance
η_o	non-dimensionalised height of the core from the central line of the rectangular channel
τ	shear stress
τ_o	yield stress

REFERENCES

1 Aris, R (1956) On the dispersion of a solute in a fluid flowing through a tube Proc Roy Soc , Series A Vol 235, p 67

2 Chum, S E and Furland, G S (1962) The flow behaviour and shear stress-shear rate characteristics of canine blood Amer J Physiology, Vol 203, p 417

3 "__________" (1965) Viscometry of human blood for shear rates of 0-100,000 s^{-1} Nature, Vol 206, p 617

4 Cokelet et al (1963) Rheology of human blood-measurement near and at zero shear rate Trans Soc Rheol , Vol 7, p 303

5 Fan, L.T and Hwang, W S. (1965) Dispersion of Ostwald de Waele fluid in laminar flow through a cylinderical tube Proc. Roy Soc , Series A, Vol 283, p 576

6 Fan, L.T and Wang, C B (1966). Dispersion of matter in non-Newtonian laminar flow through circular tubes Proc Roy Soc , Series A, Vol 292, p 203

7 Ghoshal,S.(1971) Dispersion of solutes in non-Newtonian flows through a circular tube Chem Engg. Sc , Vol 26, p. 185

8 Gupta, P S and Gupta, A S (1972) Effect of homogeneous and heterogeneous reactions on the dispersion of a solute in the laminar flow between two plates Proc Roy Soc , Series A, Vol 330, p 59

9 Merrill et al (1963) Non-Newtonian Rheology of human blood. Effect of fibrinogen deduced by subtraction Circulation Res , Vol 13, p 48

10 Shah, S N and Cox, K E. (1974) Dispersion of solutes in non-newtonian laminar flow through a circular tube Eyring model fluid. Chem Engg Sc , Vol 29, p. 1282

11 Taylor, Sir Geoffrey (1953) Dispersion of soluble matter in solvent flowing slowly through a tube Proc Roy. Soc , Series A, Vol 219, p 186

12 ________" (1954) Dispersion of matter in turbulent flow through a pipe Proc Roy Soc , Series A, Vol 223, p 446

13 ________" (1954) Conditions under which dispersion of a solute in a stream of solvent can be used to measure molecular diffusion Proc Roy Soc , Series A, Vol 225, p 473

14 Wooding, R A. (1960) Instability of viscous liquid of variable density in a vertical Hele-Show cell J Fluid Mech. Vol 7, p 501

CHAPTER VI

EFFECTS OF HOMOGENEOUS REACTION ON THE DISPERSION IN LAMINAR FLOW OF A BINGHAM FLUID

6 1 INTRODUCTION

In Chapter IV we have investigated the effects of viscosity variation on the dispersion of a solute in the presence of homogeneous and heterogeneous reactions, in a fluid of variable viscosity, flowing between parallel plates and in circular tubes, using Taylor's approach In Chapter V, we have studied the dispersion in a Casson model fluid flowing in the two geometries stated above, using Taylor's method The effects of homogeneous reaction have also been investigated

In this Chapter, we study again the effects of homogeneous reaction on dispersion of a solute in a non-Newtonian fluid flowing between parallel plates and in circular tubes by considering the Bingham model

6.2 EFFECT OF HOMOGENEOUS REACTION ON DISPERSION IN A BINGHAM PLASTIC LIQUID FLOWING BETWEEN PARALLEL PLATES

Consider the laminar flow of Bingham fluid under a constant pressure gradient between two parallel plates distant 2h apart The physical configuration and the coordinate system are shown in Fig (6 1)

The equation of motion for one dimensional steady laminar flow of Bingham fluid is given by

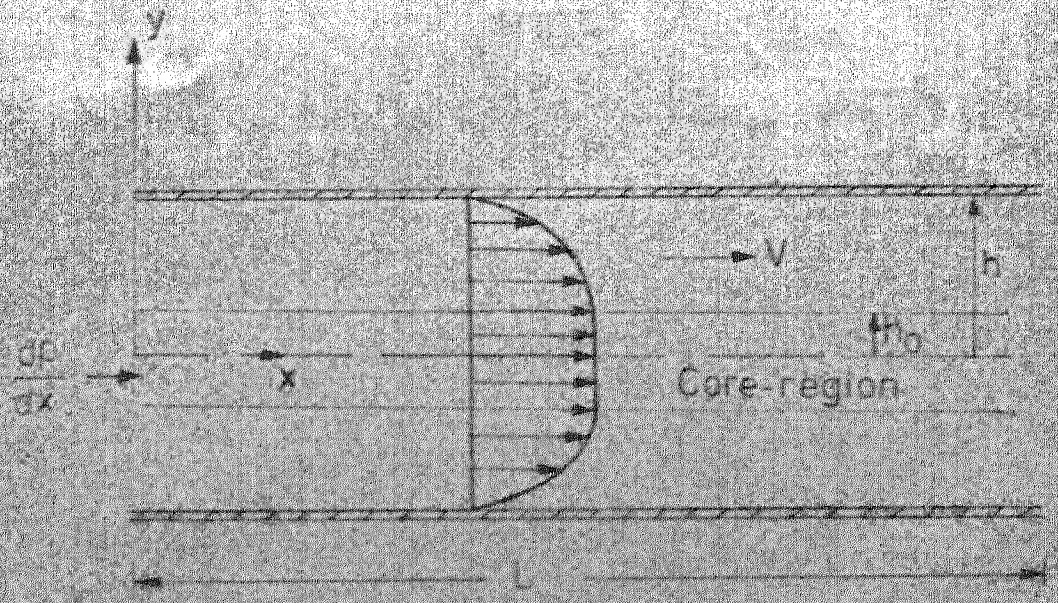

FIG. 6.1 PARALLEL PLATE GEOMETRY

$$-\frac{dp}{dx} - \frac{d\tau}{dy} = 0 \qquad (6\ 1)$$

and the constitutive equation is given by

$$\mu\left(-\frac{dV}{dy}\right) = \tau - \tau_o \qquad (\tau \geq \tau_o) \qquad (6\ 2\text{-a})$$

$$\frac{dV}{dy} = 0 \qquad (\tau \leq \tau_o) \qquad (6\ 2\text{-b})$$

where τ_o is the yield stress of the fluid and μ is the viscosity

Integrating equation (6 1) and using the boundary condition

$$\tau = 0 \text{ at } y = 0$$

we get

$$\tau = \left(-\frac{dp}{dx}\right) y \qquad (6\ 3)$$

If y_o is the height of the core, we have

$$\tau_o = \left(-\frac{dp}{dx}\right) y_o \qquad (6.4)$$

Eliminating τ from equations (6 2-a) and (6 3) and using no slip condition at the wall after integration, we obtain

$$V = \frac{1}{\mu}\left[\left(-\frac{dp}{dx}\right)\left(\frac{h^2 - y^2}{2}\right) - \tau_o(h-y)\right] \qquad (6.5)$$

in the region $(y_o \leq y \leq h)$

The velocity of the core is given by

$$V_o = \frac{1}{\mu}\left[\left(-\frac{dp}{dx}\right)\left(\frac{h^2 - y_o^2}{2}\right) - \tau_o\,(h - y_o)\right] \qquad (6.6)$$

in the region $(0 \leq y \leq y_o)$

Substituting $\eta = y/h$, $\eta_o = y_o/h$ in equations (6.6) and (6.5) and dividing we obtain

$$\frac{V}{V_c} = 1 - \frac{(\eta - \eta_o)^2}{(1 - \eta_o)^2} \qquad (\eta_o \leq \eta \leq 1) \tag{6.7-a}$$

$$\frac{V}{V_c} = 1 \qquad (0 \leq \eta \leq \eta_o) \tag{6.7-b}$$

The average velocity $\bar{V}$ which is defined by

$$\bar{V} = \frac{1}{2h} \int_{-h}^{h} V \, dy \tag{6.8}$$

is obtained as

$$\bar{V} = m \, V_c \tag{6.9}$$

where

$$m = \frac{2 + \eta_o}{3} \tag{6.10}$$

We now assume that the diffusing matter while dispersing undergoes an irreversible chemical reaction in the fluid under isothermal conditions. The diffusion equation in this case can be written as

$$\frac{\partial c}{\partial t} + V \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial y^2} - Kc \tag{6.11}$$

where D is the molecular diffusion coefficient and K is the reaction rate constant. Following Taylor (1953) as in the last two chapters, the differential equation governing the concentration of the solute relative to a plane moving with the mean speed of the flow is given by

$$\frac{\partial^2 c}{\partial \eta^2} - \alpha^2 c = \frac{h^2}{DL} \left(\frac{\partial c}{\partial \xi}\right) \quad V_x \qquad (6\ 12)$$

where

$$\alpha^2 = \frac{Fh^2}{D} \qquad (6\ 13)$$

and V_x, the velocity of the fluid relative to the moving plane is given as follows

$$V_x = \bar{V} \left[\left(\frac{1}{m} - 1\right) - \frac{(\eta - \eta_o)^2}{m(1 - \eta_o)^2}\right] \qquad (\eta_o \leq \eta \leq 1) \qquad (6\ 14\text{-a})$$

$$= \bar{V} \left(\frac{1}{m} - 1\right) \qquad (0 \leq \eta \leq \eta_o) \qquad (6\ 14\text{-b})$$

The boundary conditions are

(i) $\frac{\partial c}{\partial \eta} = 0$ at $\eta = 0$ (6 15-a)

(ii) $\frac{\partial c}{\partial \eta} = 0$ at $\eta = 1$ (6 15-b)

(iii) c and $\frac{\partial c}{\partial \eta}$ are continuous at $\eta = \eta_o$ (6 15-c)

Solving the equation (6 12) with the above boundary conditions, we obtain the concentration of the solute as follows

$$c = H\left[A_1 \cosh \alpha\eta + B_1 \sinh \alpha\eta + F(\eta)\right] \quad (\eta_o \leq \eta \leq 1) \qquad (6\ 16)$$

where

$$A_1 = \frac{-2}{m\alpha^3(1-\eta_o)^2 \sinh \alpha}\left[(1 - \eta_o) + \frac{\sinh \alpha\eta_o \cosh \alpha}{\alpha}\right] \qquad (6\ 17\text{-a})$$

$$B_1 = \frac{2 \sinh \alpha\eta_o}{m\alpha^4(1-\eta_o)^2} \qquad (6\ 17\text{-b})$$

$$F(\eta) = \frac{1}{m\alpha^2}\left[\frac{(\eta - \eta_o)^2}{(1 - \eta_o)^2} + \frac{2}{\alpha^2(1 - \eta_o)^2} - \frac{(1 - \eta_o)}{3}\right] \tag{6.18}$$

$$H = \frac{h^2 \bar{V}}{DL}\left(\frac{\partial c}{\partial \xi}\right) \tag{6 19}$$

$$c = H\left[A_2 \cosh \alpha\eta - \frac{(1 - \eta_o)}{3m\alpha^2}\right] \quad (0 \leq \eta \leq \eta_o) \tag{6 20}$$

where

$$A_2 = \frac{2}{m\alpha^3(1-\eta_o)^2 \sinh \alpha}\left[\frac{\sinh(\alpha - \alpha\eta_o)}{\alpha} - (1 - \eta_o)\right] \tag{6 21}$$

when $\eta_o = 0$, we get the case of a Newtonian fluid for which c is deduced from equation (6 16) as

$$c = \frac{h^4 (dp/d\xi)\,(\partial c/\partial \xi)}{\mu\, DL^2\, \alpha^3 \sinh \alpha} \cosh \alpha\eta$$

$$+ \frac{h^4}{2\mu\, DL^2}\left(\frac{dp}{d\xi}\right)\left(\frac{\partial c}{\partial \xi}\right)\left[\frac{1}{3} - \frac{2}{\alpha^2} - \eta^2\right] \tag{6 22}$$

which has been obtained by Gupta et al (1972)

The volumetric flow rate at which the solute is transported across a section of the channel of unit breadth is given by

$$Q = 2\int_0^{\eta_o} c\, V_x\, d\eta + 2\int_{\eta_o}^{1} c\, V_x\, d\eta \tag{6.23}$$

Substituting appropriate expressions for c and V_x from equations (6 16), (6 20), (6 14-a), and (6 14-b) and integrating, we get,

$$Q = \frac{-2h^3}{DL} \left(\frac{\partial c}{\partial \xi}\right) \frac{\bar{V}^2}{m^2 \alpha^2} \quad F(\eta_o, \alpha) \tag{6 24}$$

where

$$F(\eta_o, \alpha) = \frac{(1-\eta_o)}{5} - \frac{(1-\eta_o)^2}{9} - \frac{4}{3\alpha^2(1-\eta_o)} - \frac{4}{\alpha^4(1-\eta_o)^3}$$

$$+ \frac{4 \coth \alpha}{\alpha^3(1-\eta_o)^2} + \frac{8 \sinh(\alpha\eta_o)}{\alpha^4(1-\eta_o)^3 \sinh \alpha}$$

$$- \frac{4 \sinh(\alpha\eta_o) \sinh(\alpha - \alpha\eta_o)}{\alpha^5(1-\eta_o)^4 \sinh \alpha} \tag{6.25}$$

Comparing with Fick's law of diffusion we get the equivalent dispersion coefficient D^* as

$$D^* = \frac{h^2 \bar{V}^2}{D} \frac{1}{m^2 \alpha^2} \quad F(\eta_o, \alpha) \tag{6 26}$$

When $\eta_o = 0$, we get D^* as

$$D^*_{\eta=0} = \frac{9h^2 \bar{V}^2}{D} \frac{1}{\alpha^2} \left[\frac{1}{45} - \frac{1}{3\alpha^2} + \frac{\coth \alpha}{\alpha^3} - \frac{1}{\alpha^4}\right] \tag{6 27}$$

which coincides with the result obtained by Gupta et al (1972)

When $\alpha = 0$, i e there is no homogeneous reaction, D^* reduces to

$$D^*_{\alpha=0} = \frac{h^2 \bar{V}^2}{D} \frac{8}{105} \left(1 + \frac{33\eta_o}{16} + \frac{21\eta_o^2}{16}\right) \left(\frac{1-\eta_o}{2+\eta_o}\right)^2 \tag{6 28}$$

which has been obtained by Fan and Wang (1966)

To study the effects of homogeneous reaction on the dispersion coefficient, $F(n_o,\alpha)$ has been calculated and tabulated in table (6 1)

Table 6 1

n_o	α	$F(n_o,\alpha)$
	2 0	$0\ 13492921 \times 10^{-1}$
	4 0	$0\ 73129225 \times 10^{-2}$
0 01	6 0	$0\ 41691002 \times 10^{-2}$
	8 0	$0\ 26109522 \times 10^{-2}$
	10 0	$0\ 17664779 \times 10^{-2}$
	2 0	$0\ 12957167 \times 10^{-1}$
	4 0	$0.70345273 \times 10^{-2}$
0 05	6 0	$0\ 40168828 \times 10^{-2}$
	8.0	$0\ 25186767 \times 10^{-2}$
	10 0	$0\ 17054973 \times 10^{-2}$
	2 0	$0\ 12229157 \times 10^{-1}$
	4.0	$0.66572481 \times 10^{-2}$
0 1	6.0	$0.38111906 \times 10^{-2}$
	8.0	$0.23942817 \times 10^{-2}$
	10 0	$0.16234391 \times 10^{-2}$
	2 0	$0.10616444 \times 10^{-1}$
	4 0	$0\ 58239809 \times 10^{-2}$
0 2	6 0	$0\ 33584326 \times 10^{-2}$
	8.0	$0.21212889 \times 10^{-2}$
	10 0	$0.14437851 \times 10^{-2}$

η_o	α	$F(\eta_o,\alpha)$
	2 0	0 88650938 $\times 10^{-2}$
	4 0	0 49187589 $\times 10^{-2}$
0.3	6 0	0 28667922 $\times 10^{-2}$
	8 0	0 18251791 $\times 10^{-2}$
	10 0	0 12491804 $\times 10^{-2}$
	2 0	0 70614697 $\times 10^{-2}$
	4 0	0 39797627 $\times 10^{-2}$
0 4	6 0	0 23537860 $\times 10^{-2}$
	8 0	0 15151905 $\times 10^{-2}$
	10 0	0 10451867 $\times 10^{-2}$
	2 0	0.52917812 $\times 10^{-2}$
	4 0	0.30440483 $\times 10^{-2}$
0 5	6 0	0 18355776 $\times 10^{-2}$
	8 0	0 11993229 $\times 10^{-2}$
	10 0	0 83633334 $\times 10^{-3}$

It can be seen from the above table that for a given η_o, $F(\eta_o,\alpha)$ decreases as α increases and it decreases further as η_o increases

6 3 EFFECT OF HOMOGENEOUS REACTION ON DISPERSION IN LAMINAR FLOW OF BINGHAM FLUID IN A CIRCULAR TUBE

In this section we study the effects of homogeneous reaction on dispersion of a solute in the laminar flow of a Bingham fluid in a

circular tube The physical configuration and coordinate system are shown in Fig (6 2) The equation of motion for one dimensional steady laminar flow of Bingham fluid is given by

$$-\frac{dp}{dx} - \frac{1}{r}\frac{d}{dr}(r\tau) = 0 \qquad (6\ 29)$$

and the constitutive equation for the Bingham fluid is

$$\mu\left(-\frac{dV}{dr}\right) = \tau - \tau_o \qquad (\tau \geq \tau_o) \qquad (6\ 30\text{-a})$$

$$\frac{dV}{dr} = 0 \qquad (\tau \leq \tau_o) \qquad (6\ 30\text{-b})$$

where τ_o is the yield stress and μ is the viscosity Following the similar procedure as in the last section, the velocity profile is given by

$$V = \frac{1}{\mu}\left[\left(-\frac{dp}{dx}\right)\left(\frac{R^2-r^2}{4}\right) - \tau_o(R-r)\right] \qquad (6\ 31)$$

in the region $(r_o \leq r \leq R)$, and

$$V_o = \frac{1}{\mu}\left[\left(-\frac{dp}{dx}\right)\left(\frac{R^2-r_o^2}{4}\right) - \tau_o(R-r_o)\right] \qquad (6\ 32)$$

in the region $(0 \leq r \leq r_o)$, where r_o, the radius of the core, is given by

$$r_o = \frac{2\tau_o}{(-dp/dx)} \ . \qquad (6\ 33)$$

Substituting $\eta = r/R$ and $\eta_o = r_o/R$, we can write the velocity profile as

$$V = V_o\left[1 - \frac{(\eta - \eta_o)^2}{(1 - \eta_o)^2}\right] \qquad (\eta_o \leq \eta \leq 1) \qquad (6\ 34\text{-a})$$

$$= V_o \qquad (0 \leq \eta \leq \eta_o) \qquad (6\ 34\text{-b})$$

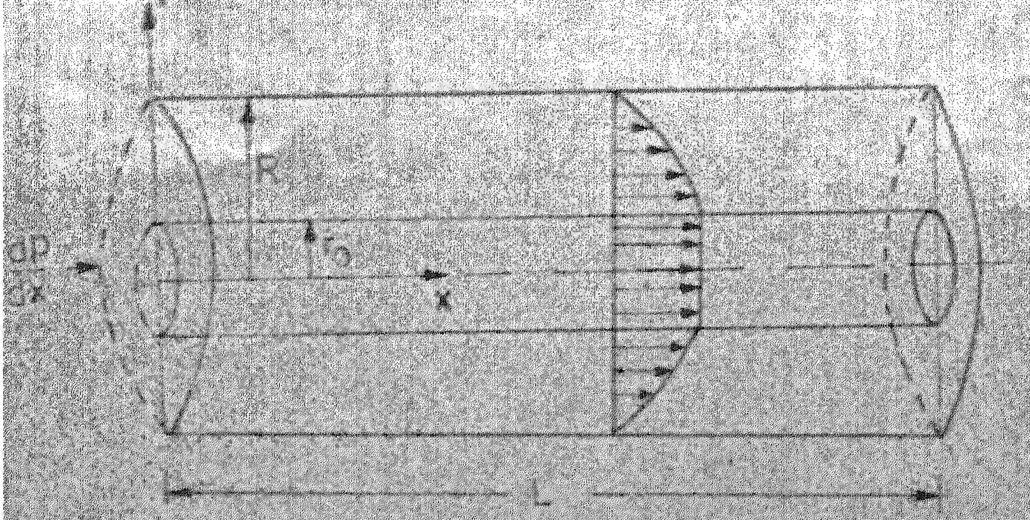

FIG.6.2 CYLINDRICAL TUBE GEOMETRY

The average velocity $\bar{V}$ is defined as

$$\bar{V} = \frac{2}{R^2} \int_0^R r\, V\, dr \tag{6 35}$$

Using the equations (6 34-a), (6 34-b) in equation (6 35) we get the average velocity as

$$\bar{V} = n\, V_c \tag{6 36}$$

where

$$n = \frac{3 + 2n_o + n_o^2}{6} \tag{6 37}$$

The velocity of the fluid relative to a plane moving with the mean speed of the flow is obtained as

$$V_x = V - \bar{V} = \bar{V}\ \left[\frac{1-n}{n} - \frac{(n - n_o)^2}{n(1 - n_o)^2}\right] \tag{6.38}$$

in the region $(n_o \leq n \leq 1)$, and

$$V_x = \bar{V}\,\left[\frac{1 - n}{n}\right] \tag{6 39}$$

in the region $(0 \leq n \leq n_o)$

As before, the diffusion equation in this case is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial c}{\partial r}\right) - K\ \ c = V \frac{\partial c}{\partial x} + \frac{\partial c}{\partial t} \tag{6 40}$$

Following Taylor (1953) the non-dimensionalised diffusion equation governing the concentration of the solute relative to a plane moving with the mean speed of the flow is obtained as

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial c}{\partial\eta}\right) - \alpha^2 c = H\left[\left(\frac{1}{r} - 1\right) - \frac{(\eta - \eta_o)^2}{\eta(1-\eta_o)^2}\right] \quad (\eta_o \le \eta \le 1) \qquad (6\ 41\text{-a})$$

$$= \frac{-(1-r)}{r} \qquad (0 \le \eta \le \eta_o) \qquad (6\ 41\text{-b})$$

where

$$H = \frac{R^2}{DL}\left(\frac{\partial c}{\partial \xi}\right) \quad \bar{V}$$

The boundary conditions are

$$(i) \quad \frac{\partial c}{\partial \eta} = 0 \quad \text{at } \eta = 0 \qquad (6\ 42\text{-a})$$

$$(ii) \quad \frac{\partial c}{\partial \eta} = 0 \quad \text{at } \eta = 1 \qquad (6.42\text{-b})$$

$$(iii) \quad c \text{ and } \frac{\partial c}{\partial \eta} \text{ are continuous at } \eta = \eta_o \qquad (6.42\text{-c})$$

Solving the equations(6.41-a) and (6 41-b) with the above boundary conditions, we obtain the concentration profile as follows

$$c = A_1\, I_o(\alpha\eta) - \frac{H(1-r)}{r\,\alpha^2} \qquad (0 \le \eta \le \eta_o) \qquad (6.43)$$

where

$$A_1 = \frac{H}{\eta(1-\eta_o)^2}\left[\left(\frac{2\eta_o^2}{\alpha} + \frac{4}{\alpha^3}\right)\eta_o\left\{\frac{K_1(\alpha\eta_o)\,I_1(\alpha) - I_1(\alpha\eta_o)K_1(\alpha)}{I_1(\alpha)}\right\}\right.$$

$$+ \frac{2\eta_o^2}{\alpha^2}\left\{\frac{I_o(\alpha\eta_o)K_1(\alpha) + I_1(\alpha)\,K_o(\alpha\eta_o)}{I_1(\alpha)}\right\}$$

$$\left. - \frac{2}{\alpha^3\, I_1(\alpha)} - \frac{2\eta_o \quad S_1}{I_1(\alpha)}\right] \qquad (6.44)$$

$$S_1 = \int_{\eta_o}^{1} [I_1(\alpha)K_o(\alpha t) + I_o(\alpha t)\,K_1(\alpha)]\ t^2\ \ dt \qquad (6.45)$$

and I_0, K_0 are modified Bessel functions of first and second kind of order zero and I_1, K_1 are of order 1, and

$$c = A_2\, I_0(\alpha\eta) + B_2\, K_0(\alpha\eta) - \frac{H}{\eta(1-\eta_0)^2}\Big[\frac{\phi}{\alpha^2} - \Big(\frac{\eta^2}{\alpha^2} + \frac{4}{\alpha^4}\Big) - 2\eta_0\, F(\eta)\Big] \quad (\eta_0 \le \eta \le 1) \tag{6.46}$$

where

$$\phi = (1-\eta)(1-\eta_0)^2 - \eta_0^2 \tag{6.47}$$

$$F(\eta) = \int_{\eta_0}^{\eta} [I_0(\alpha\eta)\, K_0(\alpha t) - I_0(\alpha t)\, K_0(\alpha\eta)]\; t^2\, dt \tag{6.48}$$

$$A_2 = \frac{-H}{\eta(1-\eta_0)^2}\Big[K_1(\alpha)\;\Big\{\Big(\frac{2\eta_0^2}{\alpha} + \frac{4}{\alpha^3}\Big)\frac{\eta_0\, I_1(\alpha\eta_0)}{I_1(\alpha)} - \frac{2\eta_0^2\, I_0(\alpha\eta_0)}{\alpha^2\, I_1(\alpha)}\Big\} + \frac{2}{\alpha^3\, I_1(\alpha)} + \frac{2\eta_0\, S_1}{I_1(\alpha)}\Big] \tag{6.49}$$

$$B_2 = \frac{-H\,\eta_0}{\eta(1-\eta_0)^2}\Big[\Big(\frac{2\eta_0^2}{\alpha} + \frac{4}{\alpha^3}\Big) I_1(\alpha\eta_0) - \frac{2\eta_0\, I_0(\alpha\eta_0)}{\alpha^2}\Big] \tag{6.50}$$

When $\eta_0 = 0$, we get the case of a Newtonian fluid for which we obtain c, from equation (6.46) by putting $\eta_0 = 0$, as

$$c = \frac{-4H\, I_0(\alpha\eta)}{\alpha^3 I_1(\alpha)} + \frac{2H\,\eta^2}{\alpha^2} + \frac{8h}{\alpha^4} - \frac{H}{\alpha^2} \tag{6.51}$$

which is the same as obtained earlier.

The volumetric rate at which the solute is transported across a section of the tube is given by

$$Q = 2\pi R^2 \int_0^{\eta_o} \eta \quad c \quad {}_x \, d\eta + 2\pi R^2 \int_{\eta_o}^{1} \eta \quad c \quad V_x \, d\eta \tag{6 52}$$

Substituting appropriate expressions for c and V_x from equations (6 43), (6 46), (6 38) and (6 39) in the equations (6 52) we obtain

$$\frac{Q}{\pi R^2} = - \frac{2R^2 \bar{V}^2}{D} \quad \left(\frac{\partial c}{\partial x}\right) \quad \left[\frac{M_1}{2n^2 \alpha^2} + \frac{M_2}{n^2(1-\eta_o)^4}\right] \tag{6 53}$$

where

$$M_1 = (1-n)^2 \eta_o^2 - \frac{2 \quad A_1(1-n) \; n \; \alpha \; \eta_o \; I_1(\alpha\eta_o)}{H} \tag{6.54}$$

$$M_2 = \frac{\phi \quad \varepsilon_1}{\alpha} \{\eta_o \; I_1(\alpha\eta_o) - I_1(\alpha)\} + \varepsilon_1 \quad \{\frac{I_1(\alpha)}{\alpha} - \frac{2I_2(\alpha)}{\alpha^2}$$

$$- \frac{\eta_o^3 \quad I_1(\alpha\eta_o)}{\alpha} + \frac{2\eta_o^2 \; I_2(\alpha\eta_o)}{\alpha^2}\} \quad + \frac{\phi \; \varepsilon_2}{\alpha} \quad \{K_1(\alpha)$$

$$- \eta_o \; K_1(\alpha\eta_o)\} \quad - \quad \varepsilon_2 \quad \{\frac{K_1(\alpha)}{\alpha} + \frac{2K_2(\alpha)}{\alpha^2}$$

$$- \frac{\eta_o^3 \; K_1(\alpha\eta_o)}{\alpha} - \frac{2\eta_o^2 \; K_2(\alpha\eta_o)}{\alpha^2}\} \quad + \frac{\phi^2 \; (1 - \eta_o^2)}{2\alpha^2}$$

$$- \frac{\phi(1 - \eta_o^4)}{2\alpha^2} - \frac{2\phi \; (1 - \eta_o^2)}{\alpha^4} + \frac{2\eta_o \; \phi(1 - \eta_o^3)}{3\alpha^2}$$

$$- \frac{2\eta_o(1-\eta_o^5)}{5\alpha^2} - \frac{8\eta_o(1-\eta_o^3)}{3\alpha^4} + \frac{(1-\eta_o^6)}{6\alpha^2} + \frac{(1-\eta_o^4)}{\alpha^4}$$

$$- 2\eta_o\,\epsilon_1 \int_{\eta_o}^{1} \eta^2\, I_o(\alpha\eta)\, d\eta - 2\eta_o\,\epsilon_2 \int_{\eta_o}^{1} \eta^2\, K_o(\alpha\eta)\, d\eta$$

$$- 2\eta_o \int_{\eta_o}^{1} (\phi\eta + 2\eta_o\,\eta - \eta^3)\, F(\eta)\, d\eta \tag{6 55}$$

and

$$\epsilon_1 = \frac{A_2 \quad \eta \quad (1-\eta_o)^2}{H} \tag{6 56}$$

$$\epsilon_2 = \frac{B_2 \quad \eta \quad (1-\eta_o)^2}{H} \tag{6 57}$$

A_1, A_2, B_2, ϕ, $F(\eta)$ are given by equations (6 44), (6 49), (6 50), (6.47) and (6.48)

Comparing with Fick's law of diffusion, we get the equivalent dispersion coefficient as

$$D^* = \frac{2R^2\bar{V}^2}{D} \left[\frac{M_1}{2\eta^2\alpha^2} + \frac{M_2}{\eta^2(1-\eta_o)^4}\right] \tag{6 58}$$

When $\eta_o = 0$, we get the case of a Newtonian fluid for which D^* is given by

$$D^*_{\eta=0} = \frac{R^2\bar{V}^2}{D} \left[\frac{32 I_2(\alpha)}{\alpha^5 I_1(\alpha)} + \frac{1}{3\alpha^2} - \frac{8}{\alpha^4}\right] \tag{6 59}$$

which is the same as obtained in Chapter (IV), equation (4.60)

To study the effects of the homogeneous reaction on the equivalent dispersion coefficient, equation (6 58) is simplified for small η_o, neglecting η_o^2 and higher powers, as

$$D^{*} = \frac{2R^{2}\,\bar{V}^{2}}{D\,n^{2}(1-n_{o})^{4}}\left[\left(\frac{4I_{2}(\alpha)}{\alpha^{5}I_{1}(\alpha)} + \frac{1}{24\alpha^{2}} - \frac{1}{\alpha^{4}}\right)\right.$$

$$- n_{o}\left\{\frac{S_{1}}{\alpha} - \frac{4S_{1}\,I_{2}(\alpha)}{\alpha} - \frac{1}{105} + \frac{1}{15\alpha^{2}} + \frac{8}{3\alpha^{4}}\right.$$

$$\left.\left. - \frac{\alpha^{2}}{2835} - \frac{4}{\alpha^{3}\,I_{1}(\alpha)}\left(\frac{1}{3} + \frac{\alpha^{2}}{20} + \frac{\alpha^{4}}{448} + \frac{\alpha^{6}}{20736}\right)\right\}\right] = \frac{2R^{2}\bar{V}^{2}}{D}\,F(n_{o},\alpha) \quad (6\ 60)$$

It is observed that as $\alpha \to 0$, equation (6 60) reduces to

$$D^{*}_{\alpha=0} = \frac{R^{2}\,\bar{V}^{2}}{72n^{2}\,(1-n_{o})^{4}}\left[\frac{3}{8} - \frac{44n_{o}}{35}\right] \quad (6\ 61)$$

which has been obtained by Fan and Wang (1966)

Equation (6 60) has been calculated and tabulated in table (6 2), from which it can be seen that the equivalent dispersion coefficient decreases as α increases and it decreases further when n_o also increases

Table 6.2

η_o	α	$F(\eta_o, \alpha)$
0.00	0.04	$0.20831111 \times 10^{-1}$
	0.1	$0.20819444 \times 10^{-1}$
	0.3	$0.20708333 \times 10^{-1}$
	0.5	$0.20486111 \times 10^{-1}$
0.04	0.04	$0.20799367 \times 10^{-1}$
	0.1	$0.20787716 \times 10^{-1}$
	0.3	$0.20676746 \times 10^{-1}$
	0.5	$0.20454806 \times 10^{-1}$
0.08	0.04	$0.20767624 \times 10^{-1}$
	0.1	$0.20755987 \times 10^{-1}$
	0.3	$0.20645159 \times 10^{-1}$
	0.5	$0.20423500 \times 10^{-1}$
0.1	0.04	$0.20751753 \times 10^{-1}$
	0.1	$0.20740123 \times 10^{-1}$
	0.3	$0.20629365 \times 10^{-1}$
	0.5	$0.20407348 \times 10^{-1}$

Here $F(\eta_o, \alpha)$ is the coefficient of $\left(\frac{2R^2V^2}{D}\right)$ in equation (6.60)

6 4 CONCLUSIONS

The effect of homogeneous reaction on dispersion of a solute in laminar flow of a Bingham fluid in rectangular channel and circular pipe has been studied by using Taylor's approach It is observed that the equivalent dispersion coefficient decreases as the reaction rate constant increases, for a given τ_o However it decreases further if τ_o increases

It may be remarked here that the non-Newtonian behaviour of the fluid decreases the Taylor's diffusion coefficient for a given mean velocity.

NOMENCLATURE

c	concentration of a solute in the flowing solvent
D	molecular diffusion coefficient
D^*	equivalent diffusion coefficient
h	height of the rectangular channel from the central line
K	reaction rate constant
L	characteristic length along the tube or channel
p	pressure
Q	flux of the solute across a section of the channel or tube
R	radius of the circular tube
r_o	radius of the core in the case of flow in circular tube
V	velocity in the non-core region
V_o	velocity of the core
y_o	height of the core from the central line
μ	viscosity of the fluid
ξ	non-dimensionalised axial length relative to the moving plane
η	non-dimensionalised length in the transverse direction
τ	shear stress
τ_o	yield stress of the fluid

REFERENCES

1 Fan, L.T and Wang, C B (1966)
Dispersion of matter in non-Newtonian laminar flow through circular tube
Proc Roy Soc , Series A, Vol 292, p 203

2 Gupta, P S and Gupta, A S (1972)
Effect of homogeneous and heterogeneous reactions on the dispersion of a solute in the laminar flow between two plates
Proc Roy Soc , Series A, Vol 330, p 59

3 Taylor, Sir Geoffrey (1953)
Dispersion of soluble matter in solvent flowing slowly through a tube
Proc Roy Soc , Series A, Vol 219, p 186